Einstein–Cartan–Dirac theory in $(1 + 2)$ -dimensions

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Abstract Einstein–Cartan theory is formulated in $(1 + 2)$ dimensions using the algebra of exterior differential forms. A Dirac spinor is coupled to gravity and the field equations are obtained by a variational principle. The space–time torsion is found to be given algebraically in terms of a quadratic spinor condensate field. Circularly symmetric, exact solutions that collapse to AdS_3 geometry in the absence of the Dirac condensate are found.

1 Introduction

Field theories in $(1 + 2)$ dimensions often provide easy ways to check ideas that are difficult to prove in actual $(1 + 3)$ dimensions. However, it is not surprising to uncover other new ideas as well that are specific to $(1 + 2)$ dimensions. Topologically massive gravity [1] or BTZ black holes [2] are some of the best known examples to the latter case. Other aspects of gravity in $(1 + 2)$ dimensions and the relevant literature may be found in Ref. [3]. An extension of the BTZ solution with torsion is discussed by Garcia et al. [4] where the field equations are derived from an action that includes topological Chern–Simons terms.

Einstein–Cartan theory [5] on the other hand is a generalization of Einstein’s theory of gravity allowing space–time (in arbitrary number of dimensions) to have torsion in addition to curvature, and relating torsion to the density of intrinsic angular momentum of matter. Initial expectation of Trautman was that the intrinsic angular momentum may influence the occurrence of singularities in gravitational collapse or cosmology but that did not turn out to be the case [6]. In this paper we couple a Dirac spinor to Einstein–Cartan gravity in $(1 + 2)$ dimensions and derive the field

equations with algebraic torsion by a variational principle. We then give a family of circularly symmetric, rotating solutions that are asymptotically AdS_3 . The intrinsic angular momentum of these solutions is determined by the quadratic scalar invariant of the Dirac spinor which may be interpreted as a bosonic condensate field. It is remarkable that in its absence the space–time geometry collapses to AdS_3 .

1.1 Notation and conventions

We specify the space–time geometry by a triplet (M, g, ∇) where M is a 3-dimensional differentiable manifold equipped with a metric tensor

$$g = \eta_{ab} e^a \otimes e^b \quad (1)$$

of signature $(- + +)$. $\{e^a\}$ is an orthonormal co-frame dual to the frame vectors $\{X_a\}$, that is, $e^a(X_b) = \delta_b^a$. A metric compatible connection ∇ can be specified in terms of connection 1-forms $\{\omega_b^a\}$ satisfying $\omega_{ba} = -\omega_{ab}$. Then the Cartan structure equations

$$de^a + \omega_b^a \wedge e^b = T^a, \quad (2)$$

$$d\omega_b^a + \omega_c^a \wedge \omega_b^c = R_b^a \quad (3)$$

yield the space–time torsion 2-forms $\{T^a\}$ and curvature 2-forms $\{R_b^a\}$, respectively. Here d denotes the exterior derivative and $\wedge$ the wedge product. We fix the orientation of space–time by choosing the volume 3-form $*1 = e^0 \wedge e^1 \wedge e^2$ where $*$ is the Hodge star map. It is possible to decompose the connection 1-forms in a unique way as

$$\omega_b^a = \hat{\omega}_b^a + K_b^a \quad (4)$$

where $\{\hat{\omega}_b^a\}$ are the zero-torsion Levi-Civita connection 1-forms satisfying

$$de^a + \hat{\omega}_b^a \wedge e^b = 0 \quad (5)$$

and $\{K_b^a\}$ are the contortion 1-forms satisfying

$$K_b^a \wedge e^b = T^a. \quad (6)$$

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The curvature 2-forms are also decomposed in a similar way:

$$R_b^a = \hat{R}_b^a + \hat{D}K_b^a + K_c^a \wedge K_b^c \quad (7)$$

with

$$\hat{D}K_b^a = dK_b^a + \hat{\omega}_c^a \wedge K_b^c - \hat{\omega}_b^c \wedge K_c^a.$$

The field equations of Einstein–Cartan theory of gravity [5] are obtained by varying the action

$$I = \int_M (\mathcal{L}_{\text{EC}} + \mathcal{L}_M) \quad (8)$$

where the Einstein–Cartan Lagrangian density 3-form

$$\mathcal{L}_{\text{EC}} = -\frac{1}{2\kappa} R_{ab} \wedge * (e^a \wedge e^b) + \lambda * 1 \quad (9)$$

with the gravitational constant κ and the cosmological constant λ , and the matter Lagrangian density 3-form $\mathcal{L}_M$. We write the infinitesimal variations as (up to a closed form)

$$\begin{aligned} \dot{\mathcal{L}} = \dot{e}^a \wedge \left(-\frac{1}{2\kappa} R^{bc} * e_{abc} + \lambda * e_a + \tau_a \right) \\ + \frac{1}{2} \dot{\omega}^{ab} \wedge \left(-\frac{1}{\kappa} * e_{abc} T^c + \Sigma_{ab} \right) \end{aligned} \quad (10)$$

where the variations of the matter Lagrangian yield the stress-energy 2-forms

$$\tau_a = \frac{\partial \mathcal{L}_M}{\partial e^a} = T_{ab} * e^b \quad (11)$$

and the angular momentum 2-forms

$$\Sigma_{ab} = \frac{\partial \mathcal{L}_M}{\partial \omega^{ab}} = S_{ab,c} * e^c. \quad (12)$$

Therefore the Einstein–Cartan field equations are given either as

$$\frac{1}{2} R^{bc} * e_{abc} - \kappa \lambda * e_a = \kappa \tau_a, \quad (13)$$

$$* e_{abc} T^c = \kappa \Sigma_{ab} \quad (14)$$

or equivalently as

$$R_{ab} = \kappa \lambda (e_a \wedge e_b) - \kappa * e_{abc} \tau^c, \quad (15)$$

$$T_a = -\frac{1}{2} \kappa * e_{abc} \Sigma^{bc}. \quad (16)$$

2 Einstein–Cartan–Dirac field equations

Let us consider a Dirac spinor field

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad (17)$$

and the conjugate spinor field

$$\bar{\psi} = \psi^\dagger \gamma_0 = (-\psi_2^*, \psi_1^*) \quad (18)$$

where ψ_1 and ψ_2 are complex, odd Grassmann valued functions. We use a real (i.e. Majorana) realization of the gamma matrices $\{\gamma_a\}$ given explicitly as

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ \gamma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \gamma_2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (19)$$

The exterior covariant derivatives of the spinor fields are defined to be

$$\nabla \psi = d\psi + \frac{1}{2} \omega^{ab} \sigma_{ab} \psi, \quad (20)$$

$$\bar{\nabla} \bar{\psi} = d\bar{\psi} - \frac{1}{2} \omega^{ab} \bar{\psi} \sigma_{ab}$$

with

$$\sigma_{ab} = \frac{1}{4} [\gamma_a, \gamma_b] = \frac{1}{2} * e_{abc} \gamma^c. \quad (21)$$

We set $*\gamma = \gamma_a * e^a$.

Next let us introduce the (Hermitian) Dirac Lagrangian density 3-form

$$\mathcal{L}_D = \frac{i}{2} (\bar{\psi} * \gamma \wedge \nabla \psi - \bar{\nabla} \bar{\psi} \wedge * \gamma \psi) + im \bar{\psi} \psi * 1. \quad (22)$$

Its infinitesimal variations are found to be (up to a closed form)

$$\begin{aligned} \dot{\mathcal{L}}_D = \dot{e}^a \wedge \left\{ \frac{i}{2} * e_a^b \wedge (\bar{\psi} \gamma_b \nabla \psi - \bar{\nabla} \bar{\psi} \gamma_b \psi) \right. \\ \left. + im \bar{\psi} \psi * e_a \right\} \\ + \frac{1}{2} \dot{\omega}^{ab} \wedge \left\{ \frac{i}{2} \bar{\psi} (* \gamma \sigma_{ab} + \sigma_{ab} * \gamma) \psi \right\} \\ + i \dot{\psi} \left\{ * \gamma \wedge \nabla \psi + \frac{1}{2} * e_b^a \wedge T^b \gamma_a \psi + m * \psi \right\} \\ - i \left\{ \bar{\nabla} \bar{\psi} \wedge * \gamma - \frac{1}{2} * e_b^a \wedge T^b \bar{\psi} \gamma_a - m * \bar{\psi} \right\} \dot{\psi}. \end{aligned} \quad (23)$$

From the above expression we identify the stress-energy 2-forms

$$\tau_a = \frac{i}{2} * e_a^b (\bar{\psi} \gamma_b \nabla \psi - \bar{\nabla} \bar{\psi} \gamma_b \psi) + im \bar{\psi} \psi * e_a \quad (24)$$

and the angular momentum 2-forms

$$\Sigma_{ab} = \frac{i}{2} (\bar{\psi} * \gamma \sigma_{ab} \psi + \bar{\psi} \sigma_{ab} * \gamma \psi). \quad (25)$$

Substituting these into the Einstein–Cartan equations (15) and (16) we obtain

$$\begin{aligned}
R_{ab} = & \kappa \lambda e_a \wedge e_b + i m \kappa \bar{\psi} \psi e_a \wedge e_b \\
& + i \frac{\kappa}{2} e_a \wedge (\bar{\psi} \gamma_b \nabla \psi) - i \frac{\kappa}{2} e_b \wedge (\bar{\psi} \gamma_a \nabla \psi) \\
& + i \frac{\kappa}{2} e_b \wedge (\nabla \bar{\psi} \gamma_a \psi) - i \frac{\kappa}{2} e_a \wedge (\nabla \bar{\psi} \gamma_b \psi), \quad (26)
\end{aligned}$$

$$T_a = i \frac{\kappa}{2} \bar{\psi} \psi * e_a \quad (27)$$

and the Dirac equation

$$*\gamma \wedge \nabla \psi + m \psi * 1 = 0. \quad (28)$$

We note that with the torsion 2-forms (27), $*e_b^a \wedge T^b = 0$ identically and the Dirac equation simplifies to (28).

These field equations can be re-written in terms of the Levi-Civita connection only [7]. To see this, we solve (27) for the contortion 1-forms

$$K_{ab} = -i \frac{\kappa}{4} \bar{\psi} \psi * (e_a \wedge e_b) \quad (29)$$

and substitute these into (15) which simplify to

$$\begin{aligned}
\hat{R}_{ab} = & \kappa \lambda e_a \wedge e_b - \kappa * e_{abc} \hat{\tau}^c \\
& + i \frac{\kappa}{4} d(\bar{\psi} \psi) \wedge * (e_a \wedge e_b) - \frac{3\kappa^2}{16} (\bar{\psi} \psi)^2 e_a \wedge e_b. \quad (30)
\end{aligned}$$

The Dirac equation (28) similarly simplifies to

$$*\gamma \wedge \bar{\nabla} \psi + m \psi * 1 + i \frac{3\kappa}{8} (\bar{\psi} \psi) \psi * 1 = 0. \quad (31)$$

It is interesting to note that the Einstein–Dirac equations (30) and (31) can be obtained from an action by zero-torsion constrained variations using the method of Lagrange multipliers [8]. To this end, we consider a modified Dirac Lagrangian density 3-form

$$\begin{aligned}
\mathcal{L}'_D = & \frac{i}{2} (\bar{\psi} * \gamma \wedge \hat{\nabla} \psi - \overline{\hat{\nabla} \psi} \wedge * \gamma \psi) \\
& + i m \bar{\psi} \psi * 1 - \frac{3\kappa}{16} (\bar{\psi} \psi)^2 * 1 \quad (32)
\end{aligned}$$

together with the constraint term

$$\mathcal{L}_{\text{constraint}} = (de^a + \omega_b^a \wedge e^b) \wedge \lambda_a \quad (33)$$

where $\{\lambda_a\}$ are the Lagrange multiplier 1-forms. The variation of the total action

$$I = \int_M (\mathcal{L}_{\text{EC}} + \mathcal{L}'_D + \mathcal{L}_{\text{constraint}})$$

with respect to the Lagrange multipliers imposes the constraint that the connection 1-forms are Levi-Civita. Then we solve the connection variation equations under this constraint for the Lagrange multiplier 1-forms as

$$\lambda_a = -\frac{i}{4} \bar{\psi} \psi e_a.$$

Substituting these in the remaining Einstein and Dirac equations gives precisely (30) and (31).

3 A circularly symmetric solution

We consider the metric

$$g = -f^2(r) dt^2 + h^2(r) dr^2 + r^2 (d\phi + a(r) dt)^2 \quad (34)$$

in plane polar coordinates (t, r, ϕ) . We use the notation and the techniques introduced in a previous paper [9]. The choice of the orthonormal basis 1-forms

$$\begin{aligned}
e^0 &= f(r) dt, & e^1 &= h(r) dr, \\
e^2 &= r(d\phi + a(r) dt), \quad (35)
\end{aligned}$$

leads to the Levi-Civita connection 1-forms

$$\hat{\omega}_1^0 = \alpha e^0 - \frac{\beta}{2} e^2, \quad \hat{\omega}_2^0 = -\frac{\beta}{2} e^1, \quad (36)$$

$$\hat{\omega}_2^1 = -\gamma e^2 - \frac{\beta}{2} e^0$$

where we defined

$$\alpha = \frac{f'}{fh}, \quad \beta = \frac{ra'}{fh}, \quad \gamma = \frac{1}{rh} \quad (37)$$

with ' denoting the derivative $\frac{d}{dr}$. On the other hand, assuming $i \frac{\kappa}{2} \bar{\psi} \psi = \tau(r)$, we calculate the contortion 1-forms

$$K_1^0 = \frac{\tau}{2} e^2, \quad K_2^0 = -\frac{\tau}{2} e^1, \quad K_2^1 = -\frac{\tau}{2} e^0. \quad (38)$$

As a first step towards a solution, we take a Dirac spinor that depends only on r and work out (28) in components as follows:

$$\begin{aligned}
\psi'_1 + \frac{h}{2} (\alpha + \gamma) \psi_1 + \frac{h}{4} (\beta + 3\tau + 4m) \psi_2 &= 0, \\
\psi'_2 + \frac{h}{2} (\alpha + \gamma) \psi_2 + \frac{h}{4} (\beta + 3\tau + 4m) \psi_1 &= 0. \quad (39)
\end{aligned}$$

We take the combinations $\psi_+ = \psi_1 + \psi_2$ and $\psi_- = \psi_1 - \psi_2$ and write a decoupled system of equations

$$\begin{aligned}
\psi'_+ + (k_1 + k_2) \psi_+ &= 0, \\
\psi'_- + (k_1 - k_2) \psi_- &= 0 \quad (40)
\end{aligned}$$

where we set

$$k_1 = \frac{h}{2} (\alpha + \gamma), \quad k_2 = \frac{h}{4} (\beta + 3\tau + 4m).$$

The formal solution to these equations are given by

$$\begin{aligned}
\psi_1 &= e^{-\int^r k_1 dr} (\xi_+ e^{-\int^r k_2 dr} + \xi_- e^{\int^r k_2 dr}), \\
\psi_2 &= e^{-\int^r k_1 dr} (\xi_+ e^{-\int^r k_2 dr} - \xi_- e^{\int^r k_2 dr}) \quad (41)
\end{aligned}$$

where ξ_+ and ξ_- are complex, odd Grassmann valued constants. It can easily be verified that

$$\tau(r) = i\kappa (\xi_-^* \xi_+ - \xi_+^* \xi_-) e^{-2\int^r k_1 dr}. \quad (42)$$

We next work out the Einstein field equations (15) that after simplifications reduce to the following system of coupled first order differential equations:

$$\begin{aligned}
\frac{\beta'}{2h} + \beta\gamma &= -\frac{\tau'}{2h} - \tau\alpha, \\
\frac{\gamma'}{h} + \frac{\beta^2}{4} + \gamma^2 + \kappa\lambda &= \frac{3\tau^2}{4} + \frac{\beta\tau}{2}, \\
\frac{\alpha'}{h} - \frac{3\beta^2}{4} + \alpha^2 + \kappa\lambda &= \frac{3\tau^2}{4} - \frac{\beta\tau}{2}, \\
\frac{\beta^2}{4} + \alpha\gamma + \kappa\lambda &= -\frac{3\tau^2}{4} - 2m\tau, \\
\frac{\beta'}{2h} + \beta\gamma &= \frac{\tau'}{2h} + \tau\gamma.
\end{aligned} \quad (43)$$

At this point, to be able to find an explicit solution, we fix a negative cosmological constant

$$\kappa\lambda = -\frac{1}{l^2} < 0$$

and restrict our attention to those cases for which

$$\gamma = \alpha = \frac{1}{rh}, \quad \tau = \beta = \frac{\beta_0}{r^2}.$$

We then integrate for the metric functions

$$f(r) = \frac{r}{l}, \quad h(r) = \frac{l}{r\sqrt{1 - \frac{2m\beta_0 l^2}{r^2} - \frac{\beta_0^2 l^2}{r^4}}}, \quad (44)$$

and

$$\begin{aligned}
a(r) &= \frac{1}{2l} \arcsin\left(\frac{m}{\sqrt{m^2 + \frac{1}{l^2}}}\right) \\
&\quad - \frac{1}{2l} \arcsin\left(\frac{m + \frac{\beta_0}{r^2}}{\sqrt{m^2 + \frac{1}{l^2}}}\right).
\end{aligned} \quad (45)$$

It now remains to integrate for the Dirac spinor and we find

$$\begin{aligned}
\psi_{\pm} &= \xi_{\pm} \frac{\ell}{r} \left| \frac{r^2}{\ell^2} - m\beta_0 + \sqrt{\frac{r^4}{\ell^4} - \frac{2m\beta_0 r^2}{\ell^2} - \frac{\beta_0^2}{\ell^2}} \right|^{\pm \frac{ml}{2}} \\
&\quad \times e^{\pm \frac{1}{2} \arcsin\left(\frac{m + \frac{\beta_0}{r^2}}{\sqrt{m^2 + \frac{1}{l^2}}}\right)}.
\end{aligned} \quad (46)$$

In order to understand the physical meaning of this solution we write down the metric

$$\begin{aligned}
g &= -\frac{r^2}{l^2} dt^2 + \frac{l^2 dr^2}{r^2(1 - \frac{2m\beta_0 l^2}{r^2} - \frac{\beta_0^2 l^2}{r^4})} \\
&\quad + r^2(d\phi + a(r)dt)^2.
\end{aligned} \quad (47)$$

Firstly we observe that in the absence of a Dirac condensate ($\beta_0 = 0$) the above metric collapses to the AdS_3 metric

$$g_0 = -\frac{r^2}{l^2} dt^2 + \frac{l^2}{r^2} dr^2 + r^2 d\phi^2. \quad (48)$$

Even when $\beta_0 \neq 0$, the metric $g \rightarrow g_0$ asymptotically as $r \rightarrow \infty$. Secondly we note a metric singularity at

$$r_c = \begin{cases} l\sqrt{m\beta_0 + \beta_0\sqrt{m^2 + \frac{1}{l^2}}}, & \beta_0 > 0, \\ l\sqrt{|\beta_0|\sqrt{m^2 + \frac{1}{l^2}} - m|\beta_0|}, & \beta_0 < 0. \end{cases} \quad (49)$$

This is a coordinate singularity as evidenced by a further calculation of the curvature scalar

$$\mathcal{R} = -\frac{6}{l^2} + \frac{4m\beta_0}{r^2} \quad (50)$$

and the quadratic curvature invariant

$$\begin{aligned}
*(R_{ab} \wedge *R^{ab}) &= \frac{6}{l^4} - \frac{8m\beta_0}{l^2 r^2} + \frac{\beta_0^2(8m^2 - \frac{4}{l^2})}{r^4} \\
&\quad + \frac{16m\beta_0^2}{r^6} + \frac{8\beta_0^4}{r^8}
\end{aligned} \quad (51)$$

that are regular at $r = r_c$. However, these curvature invariants exhibit an essential singularity at $r = 0$. Such a configuration is reminiscent of a black hole for which the essential curvature singularity at the coordinate origin is hidden behind an event horizon. A global extension of the above solution is tedious and will not be attempted here. Instead we will check the quasi-local conserved quantities associated with our solution at a distance $r > r_c$. A comprehensive discussion of the conserved quasi-local quantities for gravitating systems within the framework of general relativity may be found in [10, 11]. For calculational details in (1 + 2) dimensions we again refer to Ref. [9].

The quasi-local angular momentum is a constant:

$$J(r) = \frac{r^3}{f(r)h(r)} \frac{da}{dr} = \beta_0. \quad (52)$$

The quasi-local energy

$$\begin{aligned}
E(r) &= \frac{1}{h_0(r)} - \frac{1}{h(r)} \\
&= \frac{r}{l} - \sqrt{\frac{r^2}{l^2} - 2m\beta_0 - \frac{\beta_0^2}{r^2}} \simeq \frac{\beta_0 ml}{r}
\end{aligned} \quad (53)$$

in the limit as $r \rightarrow \infty$. Finally the quasi-local mass

$$\begin{aligned}
M(r) &= 2f(r)E(r) = J(r)a(r) \\
&= 2\frac{l^2}{r^2} - 2\frac{l^2}{r^2} \sqrt{1 - \frac{2m\beta_0 l^2}{r^2} - \frac{\beta_0^2 l^2}{r^4}} \\
&\quad + \frac{1}{2l} \arcsin\left(\frac{m}{\sqrt{m^2 + \frac{1}{l^2}}}\right) \\
&\quad - \frac{1}{2l} \arcsin\left(\frac{m + \frac{\beta_0}{r^2}}{\sqrt{m^2 + \frac{1}{l^2}}}\right) \\
&\simeq 2m\beta_0
\end{aligned} \quad (54)$$

in the limit as $r \rightarrow \infty$.

4 Conclusion

We have formulated the Einstein–Cartan–Dirac theory in $(1 + 2)$ dimensions using the algebra of exterior differential forms. We coupled a Dirac spinor to Einstein–Cartan gravity and obtained the field equations by a variational principle. The space–time torsion is given algebraically in terms of the quadratic spinor invariant associated with a Dirac condensate field. We then looked for rotating, circularly symmetric solutions. We found a particular class of solutions that possesses an essential curvature singularity at the origin $r = 0$ that is hidden behind an event horizon at some finite distance $r = r_c$ away from the origin. The mass and the intrinsic angular momentum of this configuration can be identified. Thus the space–time geometry we found exhibits the essential features of a black hole. It is remarkable that in the absence of the Dirac condensate field the metric collapses to the regular AdS_3 metric.

One of us (N.Ö.) has considered a similar model in a previous paper [12] where the space–time torsion was introduced independent of a Dirac spinor. Only the static solutions were discussed there rather than the stationary solutions as given here.

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