Quantum dynamics of a spin-1 condensate in a double-well potential

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(Received 11 May 2006; revised manuscript received 13 November 2006; published 5 February 2007)

We study quantum dynamics of a Bose-Einstein condensate of spin-1 atoms inside a double-well potential. Under the single spatial mode approximation for condensate wave functions localized within each of the two wells, we examine quantum entanglement and pseudo-spin-squeezing properties in both the Rabi and Josephson regimes. Quantum fluctuations is found to be more stable leading to robust multipartite entanglement in the lower end of the Josephson regime, or the mean field self-trapped region. Substantial multipartite entanglement is witnessed not only due to pseudo-spin-squeezing by the axis-twisting interaction but also from redistribution of mode-entangled atoms between the wells by the tunneling coupling.

DOI: [10.1103/PhysRevA.75.023605](https://doi.org/10.1103/PhysRevA.75.023605)

PACS number(s): 03.75.Gg, 03.75.Lm, 03.75.Mn

I. INTRODUCTION

The recent experimental demonstration of coherent nonlinear Josephson dynamics of an atomic Bose-Einstein condensate (BEC) in a double-well potential has stimulated much renewed interest in macroscopic quantum coherence [1]. Because of the competition between interwell tunneling and intrawell nonlinear interaction in a double-well condensate, distinct coupling regimes give rise to qualitatively different coherent dynamical behaviors [2], such as macroscopic self-trapping in the nonlinear regime [3–6], anharmonic oscillations in the intermediate regime [7–9], and Rabi oscillations in the linear regime, analogous to the system of a Josephson junction in more traditional superconductors and superfluids [10–13]. Early theoretical efforts have already been carried out in generalizing the related studies to two-component condensates [14–16]. Following the realization of condensates inside double-well potentials [17,18] and in arrays of optical potentials, several experiments have reported *in situ* measurements of linear and anharmonic tunneling dynamics [19–22].

Macroscopic quantum tunneling of magnetization (MQTM) [23–25] is a related emerging effect in the linear regime that has been actively investigated and recently observed in molecular nanomagnets of Mn_{12} -acetate and Fe_8 compounds [26,27]. Apart from its fundamental importance being at the boundary of quantum and classical realms, MQTM in large spin systems is expected to play an important technological role in the emerging field of spintronics as well as in the more futuristic goal of quantum computation. Quite recently, we have suggested a model system of a spin-1 atomic condensate [28–30], in a double-well potential for observing MQTM [31]. Spin-1 condensates are usually confined in an all-optical trap [32–34], where an atomic internal spin degree of freedom becomes available. Tunneling of condensate spin can be induced by a transverse magnetic field when it exceeds a critical value, depending on the strength of magnetic dipole-dipole interactions [15]. This paper provides further investigations of our proposed system, focusing on a quantum analysis of multipartite correlations among the collective spin variables.

Earlier studies of quantum correlations in a double-well system have almost exclusively focused on two-component condensates [35–41]. A strong motivation for such studies has been their relevance to quantum information processing and communication protocols [42,43]. More recently, spin-1 condensates have also attracted considerable attention largely due to their unique capabilities in creating entangled atomic states [44–48]. Many studies have concluded that in tunneling-coupled single [37] or two-species condensates [35], strong correlations among the atoms of different species or inside different wells can be built up, leading to significant multiparticle quantum entanglement. For symmetric atom-atom interactions as in a condensate, such multiparticle entanglement can be conveniently characterized in terms of spin squeezing [36,49–54]. In addition, spin-1 condensates can possess mode-entanglement [55–58]. Very recently, tunneling coupled spin-1 BECs have been proposed to generate maximally entangled atomic states through dynamic modifications of atomic scattering lengths [59] with Feshbach resonances [60–62]. In this paper, we examine the effect of tunneling coupling on the dynamical properties of spin squeezing and particle and mode entanglement of a spin-1 condensate inside a double-well potential.

This paper is organized as follows. First we briefly recast our formulation for a spin-1 atomic condensate inside a double-well potential, following our earlier semiclassical model system [31]. We then study the quantum correlation and entanglement properties of our model and present a systematic analysis of the quantum corrections to our earlier mean-field theory-based semiclassical results. We present numerical simulations for the quantum dynamics of our system in the various parameter regimes and conclude with a summary of our major results.

II. SPIN-1 CONDENSATE IN A DOUBLE-WELL POTENTIAL

Our model system consists of a dilute gas of alkali-metal atoms of nuclear spin $I=3/2$ (such as ^{23}Na or ^{87}Rb) trapped optically in a double-well potential at well below the critical temperature for Bose-Einstein condensation. In an optical

trap, all atomic hyperfine spin components are equally accessible. At the low collision energies available, only the lower hyperfine spin manifold of $f=I+S=1$ is accessible [28], whereas the $f=2$ manifold is essentially unpopulated. Within a mean field theory, such a condensate can be described by a three-component order parameter $\Psi=(\Psi_{+1}, \Psi_0, \Psi_{-1})$ and is called a spin-1 condensate [28]. Inside a dilute atomic gas, dominant interactions are the two-body binary encounters, which can be modeled by the effective contact potential $U(\vec{r}_1, \vec{r}_2)=\delta(\vec{r}_1-\vec{r}_2)V$ with [28,29]

$$V = \sum_{F=0,2} g_F \sum_{m=-F}^F \mathcal{P}_F. \quad (1)$$

The conservation of the hyperfine spin in a two-body s -wave collision is properly built in by restricting to diagonal terms in the total hyperfine spin $\vec{F}=\vec{F}_1+\vec{F}_2$. In the above rotationally invariant form, $g_F=4\pi\hbar^2 a_F/M$ and M is the atomic mass. a_F is the s -wave scattering length in the channel of total hyperfine spin F , and $\mathcal{P}_F$ is the corresponding projection operator

$$\mathcal{P}_F = \sum_{m=-F}^F O_{Fm}^\dagger O_{Fm}, \quad (2)$$

with

$$O_{Fm} = \sum_{m_1 m_2} \langle Fm | F_1=1, m_1; F_2=1, m_2 \rangle \times \langle F_1=1, m_1; F_2=1, m_2 |. \quad (3)$$

$\langle Fm | F_1=1, m_1; F_2=1, m_2 \rangle$ denotes the Clebsch-Gordon coefficient. The available total hyperfine spin channels are $F=0, 2$ for s -wave scattering between two identical bosonic atoms. Using the operator identities $\mathcal{P}_0+\mathcal{P}_2=1$ and $\vec{F}_1 \cdot \vec{F}_2 = \mathcal{P}_2 - 2\mathcal{P}_0$, the effective interaction can be expressed in a form

$$V = c_0 + c_2 \vec{F}_1 \cdot \vec{F}_2, \quad (4)$$

where $c_0=(g_0+2g_2)/3$, and $c_2=(g_2-g_0)/3$. For the two types of atoms considered, c_0 is always positive, while c_2 could be either positive for antiferromagnetic (^{23}Na) or negative for ferromagnetic (^{87}Rb) interactions. The model system is then described by

$$H = \int d^3\vec{r} \left(\frac{\hbar^2}{2M} \vec{\nabla} \hat{\Psi}_m^\dagger \cdot \vec{\nabla} \hat{\Psi}_m + V_{\text{trap}}(\vec{r}) \hat{\Psi}_m^\dagger \hat{\Psi}_m + \frac{1}{2} c_0 \hat{\Psi}_m^\dagger \hat{\Psi}_{m'}^\dagger \hat{\Psi}_{m'} \hat{\Psi}_m + \frac{1}{2} c_2 \hat{\Psi}_m^\dagger \hat{\Psi}_{m'}^\dagger \vec{F}_{mk} \cdot \vec{F}_{m'k'} \hat{\Psi}_{k'} \hat{\Psi}_k \right), \quad (5)$$

where summation over repeated indices is implied. The optical trap potential V_{trap} is assumed to be independent of the atomic hyperfine spin. The component of $\vec{F}$, $F_{x,y,z}$ denotes the respective $F=1$ spin matrices.

If the barrier separating the double-wells is not too low, we can ignore the effect of nonlinear interactions within the barrier region and assume a linear superposition for the total

field operators of the two subsystem within each well (labeled as 1 and 2), i.e., $\hat{\Psi}_m = \hat{\Psi}_{1m} + \hat{\Psi}_{2m}$, ($m=+1, 0, -1$). Both types of alkali-metal atoms we consider here satisfy $c_0 \gg |c_2|$. This allows for a simple approximate description of each component with the same single spatial wave function or the use of the single spatial mode approximation (SMA) [30], whose validity has been discussed in detail in recent literature [30,36,63–67]. Accordingly, we write $\hat{\Psi}_{im} = \hat{a}_{im}(t) \Phi_i(\vec{r}, t)$, where $\hat{a}_{1m}$ and $\hat{a}_{2m}$, respectively, describe the condensate mode annihilation operators within each of the two wells. They commute with each other under the strong barrier assumption, i.e., they obey the usual bosonic commutation rules describing the Weyl-Heisenberg algebra. The single spatial wave functions $\Phi_i(\vec{r}, t)$ are determined from the solutions the Gross-Pitaevskii equation (GPE)

$$\left[-\frac{\hbar^2}{2M} \nabla^2 + V_i(\vec{r}) + c_0(N_i - 1) |\Phi_i|^2 \right] \Phi_i = \mu_i \Phi_i, \quad (6)$$

where V_i is the spin-independent local potential describing the i th well, μ_i is the chemical potential. For simplicity, we shall consider only symmetric double-well potentials here. $\hat{N}_i = \sum_m \hat{a}_{im}^\dagger \hat{a}_{im}$ is the number operator for the i th well. We caution that, rigorously speaking, SMA based on Eq. (6) can be used only when $N_i = \langle \hat{N}_i \rangle$ is not varying in time. Otherwise, for dynamically changing population distributions, the use of the SMA will require an additional assumption that the two spatial modes do not change appreciably with population oscillations, a situation clearly valid in the self-trapped region. As an alternative approximation, in particular, often used for the conventional Josephson junction systems, we can adopt Φ_i directly from the usual Schrodinger equation for a single atom,

$$\left[-\frac{\hbar^2}{2M} \nabla^2 + V_i(\vec{r}) \right] \Phi_i = \mu_i \Phi_i, \quad (7)$$

with the corresponding validity of the approximation limited to a small number of atoms [4,35,41,68–71]. A more detailed discussion on the validity of different approximations will be presented in Sec. V. This later case is sometimes called two-mode approximation and is often used in studies of tunneling coupled single- or two-component condensates despite that an effective model system would consist of four-mode operators for a two-component condensate. We note that the recent observation of tunneling-coupled BECs for technical reasons is limited to both low temperatures and low densities, the same as what is normally required for the single spatial mode approximation. These conditions are crucial for observing many oscillations of the Josephson-like dynamics aided by the requirements of enhanced atomic detection sensitivity and essentially free from dissipative collisions [1]. More specific discussions on the validity of SMA will be discussed in the context when results and discussions are presented. Adopting the same approach, our three-component model is effectively described by six-mode operators [31,59]

$$H = H_J + H_s + H_a, \quad (8)$$

where

$$H_J = -J \sum_m (\hat{a}_{1m}^\dagger \hat{a}_{2m} + \hat{a}_{2m}^\dagger \hat{a}_{1m}), \quad (9)$$

$$H_s = \frac{u_0}{2} (\hat{N}_1^2 + \hat{N}_2^2), \quad (10)$$

$$H_a = \frac{u_2}{2} (\hat{L}_1^2 + \hat{L}_2^2), \quad (11)$$

where $u_{0,2} = c_{0,2} \int d^3r |\Phi|^4$, and a constant term $(\mu - u_2 - u_0/2)\hat{N}$ has been omitted, as it does not affect the dynamics because $[\hat{N}, H] = 0$. H_J describes the tunnel coupling between the condensates in the double wells. $H_{s,a}$, respectively, describes the symmetric and asymmetric collisional interactions under exchanges of spin indices. The angular momentum representation for collective spin variables, which is found to be convenient in the description of spin-1 condensates, will be employed here as well [30,72]. The effective angular momentum operators are introduced as $\hat{L}_i^{(z)} = \hat{N}_{i,+1} - \hat{N}_{i,-1}$, and $\hat{L}_j^{(\pm)} = \hat{L}_{jx} \pm i\hat{L}_{jy}$. They obey the usual su(2) angular momentum algebra $[\hat{L}_j^{(+)}, \hat{L}_i^{(-)}] = 2\delta_{ij}\hat{L}_j^{(z)}$, and $[\hat{L}_j^{(z)}, \hat{L}_i^{(\pm)}] = \pm\delta_{ij}\hat{L}_j^{(\pm)}$, where

$$\hat{L}_j^{(+)} = \sqrt{2}(\hat{a}_{j,+1}^\dagger \hat{a}_{j0} + \hat{a}_{j0}^\dagger \hat{a}_{j,-1}), \quad (12)$$

and $\hat{L}_j^{(-)} = \hat{L}_j^{(+)\dagger}$.

III. MULTI-PARTITE SPIN CORRELATION AND ENTANGLEMENT

In a Josephson coupled scalar condensate, the dynamical variable of interest is usually the population imbalance between the two wells, often described by the z component of a pseudospin variable $S_z = (N_1 - N_2)/2$. An initially symmetric configuration with $S_z = 0$ remains stationary or does not evolve dynamically in a symmetric double-well system. On the other hand, a spin-1 condensate inside a double well contains richer dynamics, as we have shown earlier in a mean field treatment [31]. More specifically, even for the symmetric initial configuration, there still exists dynamical evolutions in the $L_{1,2z}$, or oscillations of local magnetization, which should be contrasted with the density or population oscillations as described by an unbalanced S_z . For some initial configurations, these two types of modes are separable [31]. Generally, one would expect spin-density correlations can occur in both modes. Similar to the usual Josephson coupled systems, the density mode can be characterized in terms of the pseudospin operators

$$S_z = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2), \quad (13)$$

$$S_x = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1), \quad (14)$$

$$S_y = \frac{1}{2i}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1), \quad (15)$$

with $\hat{a}_j^\dagger = (\hat{a}_{j,+}, \hat{a}_{j0}, \hat{a}_{j,-})$ a row vector notation. This allows us to examine quantum correlations in terms of purely spin-spin correlations in addition to spin-density correlations. In this picture, the Hamiltonian looks like

$$H = -2JS_x + u_0 S_z^2 + \frac{u_2}{2} (L_1^2 + L_2^2). \quad (16)$$

We observe that $L_1^2 + L_2^2 = N^2/2 + N + 2S_z^2 - (A_1^\dagger A_1 + A_2^\dagger A_2)$, where $A_i = a_{i0}^2 - 2a_{i,+1}a_{i,-1}$ is the spin singlet annihilation operator from two spin-1 atoms. Neglecting the constant terms, our model Hamiltonian becomes

$$H = -2JS_x + (u_0 + u_2)S_z^2 - u_2(A_1^\dagger A_1 + A_2^\dagger A_2). \quad (17)$$

The last two terms of this Hamiltonian from atom-atom interaction behave as generators of pseudo-spin-squeezing. Their resulting quantum correlations properties, however, are different. The first term is a simple rotation operator. It simply rotates the uncertainty ellipsoid (or error distribution) generated from the two nonlinear terms. Though, it cannot generate quantum correlations required for entanglement, it redistributes the noise among different directions. The second term creates multiparticle quantum entanglement between the particles in different wells via pseudo-spin-squeezing through a single axis twisting operation [36,49]. The last term leads to quantum entanglement between the particles in the same well via two-spin squeezing that also may be associated with mode-entanglement [73]. A simple way to appreciate the above is to note that the last term involves spin mixing interaction in the form $(a_{i0}^2 a_{i+}^\dagger a_{i-}^\dagger + \text{H.c.})$, which is a two-mode squeezing interaction for large numbers of n_{i0} , analogous to an optical parametric oscillator process where pairs of photons in different electromagnetic modes are generated. Alternatively by identifying $U_{i+} = a_{i+}^\dagger a_{i0}$ and $V_{i+} = a_{i-}^\dagger a_{i0}$ as the spin raising operators for the U, V subspin groups of su(3) in the Gel-Mann representation [73], one may also characterize this interaction as a Barnett-Dupertuis type spin-spin interaction [74], which may be called a two-spin squeezing interaction [73]. The first term of the Hamiltonian (17) corresponds to quantum tunneling that would distribute mode entangled particles between the wells. The last two terms commute with each other, and each represents an independent source of different type of quantum entanglement in the model system. As a result, mode entanglement can be transferred to multipartite entanglement between the wells and contribute to pseudo-spin-squeezing.

When $u_2 = 0$, our model Hamiltonian becomes a generic collective spin model, similar to the usual tunneling coupled single-component (scalar) condensate in a double-well potential, where exact solutions for small atom numbers as well as semiclassical results for large atom numbers have elucidated spin squeezing and the associated multiparticle entanglement [4,37,41,70,71]. However, the formal similarity between the Hamiltonians and the spin operator algebra does not necessarily lead to equivalent states and dynamics due to

the different Hilbert spaces. For tunneling coupled condensates, a one-to-one correspondence between Fock states of atoms and the angular momentum basis states exists. Such a correspondence is, on the other hand, unknown for spinor condensates whose exact solutions are unknown. We will illustrate this point with a simple example to show the differences in Hilbert spaces and its influence on system dynamics. If atoms are symmetrically distributed in both wells so that half of them are in one well and the other half is in the other well, then we find $\langle S_x \rangle \neq 0$ for a scalar condensate if atoms in both wells are in coherent states. For a spinor condensate of $F=1$, however, the atoms in one well can be in internal state ($m=+1$), while the atoms in the other well with ($m=-1$), leading to $\langle S_x \rangle=0$. Similar differences occur if one considers higher-order moments of the operators. More generally, when u_2 cannot be ignored, the formal similarity ends. In a system of tunneling coupled spinor condensates, both single-axis twisting and two-spin (or two-mode) squeezing are present in the system [73]. This leads to substantial quantum entanglement among particles that can be characterized by the spin squeezing criterion, which is directly measurable via the Ramsey spectroscopy [50]. Our numerical results presented below shall demonstrate some of the above-mentioned features.

IV. QUANTUM CORRECTIONS TO THE MEAN FIELD THEORY

In the standard mean field theory, one treats the operators $\hat{a}_{im}$ as c numbers, which breaks the global phase symmetry in the system associated with the conservation of the total number of atoms. Quantum mechanically, it is not entirely obvious how to interpret the phase of the c number corresponding to a_{im} , as there is no phase operator conjugate to $\hat{N}$, a well-known problem in quantum optics [75], more serious for bosons with zero rest mass, such as photons for which the spin operator is not well defined. Atomic number and phase fluctuations in a BEC can also be described via massless bosons of the Goldstone mode [76]. In contrast, there exists no quantum phase problem for the angular momentum operators. The mathematical reason is simple; the number operator has a semibound spectrum (cannot be negative), whereas J_z has an unbounded one. An improved mean field treatment would therefore benefit from this fact by considering c number replacements of angular momentum operators. The dynamics of expectations for the angular momentum operators is then described by a Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy of equations of motion where the n th-order moments depend on the $(n+1)$ th-order moments [77,78]. Treating angular momentum operators as c numbers is equivalent to the lowest order truncation of this hierarchy. Further improvements can be systematically handled perturbatively by higher-order truncation approximations. Pseudo-spin-squeezing is associated with second-order correlation functions, and thus, we would have to consider truncations at least to the second order for our studies.

The operators that we use to describe the quantum Josephson dynamics are the angular momentum operators

$\hat{S}_{x,y,z}$, and the spin-singlet operators of each well $A_{1,2}$. $\hat{S}_{x,y,z}$ operators obey the $su(2)$ angular momentum algebra

$$[S_z, S_{\pm}] = \pm S_{\pm}, \quad [S_+, S_-] = 2S_z, \quad (18)$$

where $S_{\pm} = S_x \pm iS_y$. Note that $[S_{x,y,z}, N] = 0$ as $[S_x, N_j] = (-1)^j S_y$, $[S_y, N_j] = -(-1)^j S_x$, and $[S_z, N_j] = 0$, with $j=1, 2$. For the singlet operators, we have

$$[A_j, A_j^\dagger] = 4N_j + 6, \quad [A_j, N_j] = 2A_j. \quad (19)$$

These two algebra are connected to each other as follows:

$$[A_j, S_x] = -A_0, \quad [A_j, S_y] = -(-1)^j iA_0,$$

$$[A_j, S_z] = -(-1)^j A_j, \quad (20)$$

where $A_0 = a_{1,+1}a_{2,-1} + a_{1,-1}a_{2,+1} - a_{1,0}a_{2,0}$ is the spin singlet annihilation operator for two (distinguishable) spin-1 atoms from the two wells. It commutes with S_z but not with $S_{x,y}$. For convenience, we introduce

$$A_{\pm} = \frac{A_1 \pm A_2}{2}. \quad (21)$$

This simplifies notation as $[A_0, S_x] = -A_+$, and $[A_0, S_y] = -iA_-$. Using these relations and the model Hamiltonian, we derive the Heisenberg equations of motion

$$\dot{S}_z = -2JS_y,$$

$$\dot{S}_y = 2JS_z + u\{S_z, S_x\} + u_2(\{A_{0x}, A_{-x}\} + \{A_{0y}, A_{-y}\}),$$

$$\dot{S}_x = -u\{S_y, S_z\} - u_2(\{A_{+x}, A_{0y}\} - \{A_{+y}, A_{0x}\}),$$

$$\dot{A}_{0x} = 2JA_{+y} + 2u_2A_{0y} + u_2(\{S_x, A_{+y}\} + \{S_y, A_{-x}\}),$$

$$\dot{A}_{0y} = -2JA_{+x} - 2u_2A_{0x} + u_2(\{S_y, A_{-y}\} - \{S_x, A_{+x}\}),$$

$$\dot{A}_{-x} = -u_2A_{-y} + u_0\{S_z, A_{+y}\} - \frac{u_2}{2}\{N, A_{-y}\},$$

$$\dot{A}_{-y} = u_2A_{-x} - u_0\{S_z, A_{+x}\} + \frac{u_2}{2}\{N, A_{-x}\},$$

$$\dot{A}_{+x} = 2JA_{0y} + 3u_2A_{+y} + u_0\{S_z, A_{-y}\} - \frac{u_2}{2}\{N, A_{+y}\},$$

$$\dot{A}_{+y} = -2JA_{0x} - 3u_2A_{+x} - u_0\{S_z, A_{-x}\} + \frac{u_2}{2}\{N, A_{+x}\},$$

where $\hbar=1$ is assumed and $u=u_0+u_2$. For an operator $\hat{O}$, we define $\hat{O}_x = (1/2)(O+O^\dagger)$, and $O_y = (1/2i)(O-O^\dagger)$. The anti-commutator of two operators O_1 and O_2 is denoted by $\{O_1, O_2\} = O_1O_2 + O_2O_1$. Such a symmetric operation always gives Hermitian operators if O_j s are Hermitian. The expectation values of these equations yield an infinite BBGKY hierarchy of equations of motion with the dynamics of $\langle O \rangle$

depending on $\langle\{O_1, O_2\}\rangle$, whose dynamics in turn depends on $\langle\{O_1, O_2, O_3\}\rangle$, and so on. We replace $\langle\{O_1, O_2\}\rangle$ by the second-order quantum correlation function $\langle O_1, O_2 \rangle$ defined as $\langle O_1, O_2 \rangle = \langle\{O_1, O_2\}\rangle - 2\langle O_1 \rangle \langle O_2 \rangle$. In this case, the equations for the first-order moments $\langle O \rangle$ can be written as

$$\begin{aligned}
 \dot{S}_z &= -2JS_y, \\
 \dot{S}_y &= 2JS_z + u(\langle S_z, S_x \rangle + 2S_z S_x) + u_2(\langle A_{0x}, A_{-x} \rangle + 2A_{0x} A_{-x} \\
 &\quad + \langle A_{0y}, A_{-y} \rangle + 2A_{0y} A_{-y}), \\
 \dot{S}_x &= -u(\langle S_y, S_z \rangle + 2S_y S_z) - u_2(\langle A_{+x}, A_{0y} \rangle + 2A_{+x} A_{0y} \\
 &\quad - \langle A_{+y}, A_{0x} \rangle - 2A_{+y} A_{0x}), \\
 \dot{A}_{0x} &= 2JA_{+y} + 2u_2 A_{0y} + u_2(\langle S_x, A_{+y} \rangle + 2S_x A_{+y} \\
 &\quad + \langle S_y, A_{-x} \rangle + 2S_y A_{-x}), \\
 \dot{A}_{0y} &= -2JA_{+x} - 2u_2 A_{0x} + u_2(\langle S_y, A_{-y} \rangle + 2S_y A_{-y} \\
 &\quad - \langle S_x, A_{+x} \rangle - 2S_x A_{+x}), \\
 \dot{A}_{-x} &= -u_2 A_{-y} + u_0(\langle S_z, A_{+y} \rangle + 2S_z A_{+y}) \\
 &\quad - \frac{u_2}{2}(\langle N, A_{-y} \rangle + 2NA_{-y}), \\
 \dot{A}_{-y} &= u_2 A_{-x} - u_0(\langle S_z, A_{+x} \rangle + 2S_z A_{+x}) + \frac{u_2}{2}(\langle N, A_{-x} \rangle + 2NA_{-x}), \\
 \dot{A}_{+x} &= 2JA_{0y} + 3u_2 A_{+y} + u_0(\langle S_z, A_{-y} \rangle + 2S_z A_{-y}) \\
 &\quad - \frac{u_2}{2}(\langle N, A_{+y} \rangle + 2NA_{+y}), \\
 \dot{A}_{+y} &= -2JA_{0x} - 3u_2 A_{+x} - u_0(\langle S_z, A_{-x} \rangle + 2S_z A_{-x}) \\
 &\quad + \frac{u_2}{2}(\langle N, A_{+x} \rangle + 2NA_{+x}), \tag{22}
 \end{aligned}$$

where we have used shorthand notation $O \equiv \langle O \rangle$. Quantum corrections $\langle O_1, O_2 \rangle$ contribute to decoherence in the system, destroying the otherwise unitary mean field evolution, obtained by neglecting $\langle O_1, O_2 \rangle$ terms.

A truncation of the BBGKY hierarchy beyond the mean field case, in which $\langle O_1 O_2 \rangle = O_1 O_2$, can be treated by assuming [77],

$$\begin{aligned}
 \langle\{O_1, O_2, O_3\}\rangle &\approx 2(\langle O_1 \rangle \langle\{O_2, O_3\}\rangle + \langle O_2 \rangle \langle\{O_1, O_3\}\rangle \\
 &\quad + \langle O_3 \rangle \langle\{O_1, O_2\}\rangle - 4\langle O_1 \rangle \langle O_2 \rangle \langle O_3 \rangle). \tag{23}
 \end{aligned}$$

With this systematic approximation scheme for our nine operators required to characterize the dynamics of the population imbalance in the double-well system, it is necessary to include 81 additional equations for the second-order correla-

tion functions. Only 45 of these equations are needed due to symmetry considerations because of the Hermitian nature of all operators. Among these 45 equations, nine are for the self-correlations (variance) of the operators. For some initial conditions the number of relevant equations can be further reduced. To illustrate the general form of these equations, we explicitly give two of them as follows:

$$\begin{aligned}
 \frac{d}{dt} \langle S_z, S_x \rangle &= -2J \langle S_y, S_x \rangle - 2u(S_y \langle S_x, S_z \rangle + S_x \langle S_y, S_z \rangle) \\
 &\quad - 2u_2(A_{+x} \langle A_{0y}, S_z \rangle + A_{0y} \langle A_{+x}, S_z \rangle - A_{+y} \langle A_{0x}, S_z \rangle \\
 &\quad - A_{0x} \langle A_{+y}, S_z \rangle), \\
 \frac{d}{dt} \langle S_y, S_x \rangle &= 2J \langle S_z, S_x \rangle + 2u(S_z \langle S_x, S_y \rangle + S_x \langle S_0, S_x \rangle - S_y \langle S_y, S_x \rangle \\
 &\quad - S_x \langle S_y, S_y \rangle) + 2u_2(A_{0x} \langle A_{-x}, S_x \rangle + A_{-x} \langle A_{0x}, S_x \rangle \\
 &\quad + A_{0y} \langle A_{-y}, S_y \rangle - A_{-y} \langle A_{0y}, S_x \rangle) - 2u_2(A_{+x} \langle A_{0y}, S_y \rangle \\
 &\quad + A_{0y} \langle A_{+x}, S_y \rangle + A_{+y} \langle A_{0x}, S_y \rangle - A_{0x} \langle A_{+y}, S_y \rangle), \tag{24}
 \end{aligned}$$

where we have used a shorthand notation $O \equiv \langle O \rangle$. The equations for the second-order moments could be generated from a reduced set when the second-order moments may be factorized as $\langle O_i, O_j \rangle^2 = \langle O_i, O_i \rangle \langle O_j, O_j \rangle$ [77,78]. Let us introduce a shorthand notation such that $(\delta O)^2 = \langle O, O \rangle$. By the definition of second-order moments, this can be expressed in terms of the variance $\langle O, O \rangle = 2[\Delta O]^2$ so that $(\delta O)^2 = 2[\Delta O]^2$ and $\langle O_i, O_j \rangle = (\delta O_i)(\delta O_j)$. We note that factorization of second-order moments in terms of variances, $\langle O_i, O_j \rangle \propto (\Delta O_i)(\Delta O_j)$, persists in time. Dynamics of all second-order moments is then determined through the dynamics of self-correlations $(\delta O)^2$. Equations of motions for δO can be simply and directly constructed because they are equivalent to perturbations of mean field equations ($O \rightarrow O + \delta O$) [77]. We find that δO satisfy

$$\begin{aligned}
 \delta \dot{S}_z &= -2J \delta S_y, \\
 \delta \dot{S}_y &= 2J \delta S_z + 2u(\delta S_z S_x + S_z \delta S_x) + 2u_2(\delta A_{0x} A_{-x} + A_{0x} \delta A_{-x}) \\
 &\quad + 2u_2(\delta A_{0y} A_{-y} + A_{0y} \delta A_{-y}), \\
 \delta \dot{S}_x &= -2u(\delta S_y S_z + S_y \delta S_z) - 2u_2[(\delta A_{+x} A_{0y} + A_{+x} \delta A_{0y}) \\
 &\quad - (\delta A_{+y} A_{0x} + A_{+y} \delta A_{0x})], \\
 \delta \dot{A}_{0x} &= 2J \delta A_{+y} + 2u_2 \delta A_{0y} + 2u_2[(\delta S_x A_{+y} + S_x \delta A_{+y}) \\
 &\quad + (\delta S_y A_{-x} + S_y \delta A_{-x})], \\
 \delta \dot{A}_{0y} &= -2J \delta A_{+x} - 2u_2 \delta A_{0x} + u_2[(\delta S_y A_{-y} + S_y \delta A_{-y}) \\
 &\quad - (\delta S_x A_{+x} + S_x \delta A_{+x})], \\
 \delta \dot{A}_{-x} &= -u_2(1 + N) \delta A_{-y} + 2u_0(\delta S_z A_{+y} + S_z \delta A_{+y}) - u_2 \delta N A_{-y},
 \end{aligned}$$

$$\begin{aligned}
\dot{\delta A}_{-y} &= u_2(1+N)\delta A_{-x} - 2u_0(\delta S_z A_{+x} + S_z \delta A_{+x}) + u_2 \delta N A_{-x}, \\
\dot{\delta A}_{+x} &= 2J\delta A_{0y} + u_2(3-N)\delta A_{+y} + 2u_0(\delta S_z A_{-y} + S_z \delta A_{-y}) \\
&\quad - u_2 \delta N A_{+y}, \\
\dot{\delta A}_{+y} &= -2J\delta A_{0x} - u_2(3-N)\delta A_{+x} - 2u_0(\delta S_z A_{-x} + S_z \delta A_{-x}) \\
&\quad + u_2 \delta N A_{+x}.
\end{aligned} \tag{25}$$

They resemble (and may be considered equivalent in a sense [77]) the usual Bogoliubov equations. They, however, represent dynamics of second-order self-correlation functions. Factorization allows for a linearization-like treatment with respect to second-order self-correlations. Out of them, we can construct equations of motions for the second-order moments, corresponding to mutual correlations of different operators. One could immediately check, for example,

$$\begin{aligned}
\frac{d}{dt}\langle S_z, S_x \rangle &= \dot{\delta S}_z \delta S_x + \delta S_z \dot{\delta S}_x, \\
\frac{d}{dt}\langle S_y, S_x \rangle &= \dot{\delta S}_y \delta S_x + \delta S_y \dot{\delta S}_x,
\end{aligned} \tag{26}$$

which also verifies persistence of factorization. For our purposes, we shall only need self-correlation functions or variances to determine squeezing properties of collective spin variables.

In this case, the dynamics of second-order quantum corrections is governed by 18 equations, a property not shared for more general initial conditions. In this paper, we further simplify the equations by limiting our studies to short time dynamics, when the phase space trajectories remain close to the surface defined by the mean field solution so that the Bogoliubov corrections remain small. As such, we consider effects of quantum fluctuations through the dynamics of perturbation δO_j as the quantum noise to the mean field solution. In order to estimate for how long such a linearized treatment is applicable, we also consider the case where Bogoliubov back reaction terms are not neglected. Within the factorization ansatz, the second-order equations represented by the reduced set are now coupled to the mean field equations by the back reaction terms.

A similar approach has been previously employed to investigate the dephasing of Josephson oscillations between two coupled condensates [79]. We first solve equations of motion treating the operators as c numbers. Using the zeroth-order solution with respect to fluctuations, we then solve the set of coupled first-order equations in the fluctuating parts. The zeroth-order equations are analogous to the coupled Gross-Pitaevski equations (GPEs), whereas the latter, first-order equations are analogous to the Bogoliubov-de Gennes equations (BdGEs). For our model system at hand, the equations are, however, not for the wave functions. They are expressed in the second quantized form already incorporating the bosonic algebra. Thus, the zeroth order is already an expectation value, or equivalently of second order in the usual coupled GPEs for the wave functions. Each set of equations then consists of nine coupled equations, which are

numerically integrated by using a standard Runge-Kutta algorithm. To estimate the time range for the linearized treatment, the set of coupled 18 Bogoliubov back-reaction equations are also solved. Our numerical results are reported and discussed below.

V. RESULTS AND DISCUSSIONS

We focus on two dynamical regimes in this paper. Taking $J = \kappa N(u_0 + u_2)$, we identify the Rabi regime by the condition $2\kappa \gg 1$. In this regime, complete oscillations between the two wells are expected to occur. In the lower end of the Josephson regime for which $2\kappa \ll 1$, a qualitatively different behavior of self-trapping can occur. We will take $\kappa = 2$ and $\kappa = 0.02$ for studying dynamics in the Rabi regime and in the (lower end of the) Josephson regime, respectively.

Initially, all the N atoms are assumed to be in the spin-0 state located in one of the well so that

$$\Psi(0) = |0, \alpha, 0\rangle |0, 0, 0\rangle. \tag{27}$$

Though it does not yield a perfect factorization for a few correlation functions, initially, the coherent state assumption simplifies the evaluation of the initial δO_i . Such an initial state provides a reasonable approximation to the quantum noise properties inside a condensate as realized in experiments. For $N \gg 1$, the factorization becomes approximately true. Here $\delta N = \sqrt{N/2}$ remains the same at all times, as N and δN are integrals of motion. We have numerically examined regimes for larger and smaller N values, and qualitatively, the same results were found with the only significant quantitative change being the N -dependent time scale, the period of oscillations grows with decreasing N . Similar behaviors are found at shorter time that scales as N grows. Assuming $|\alpha|^2 = N$, we can employ variables scaled by N . The associated quantum fluctuations are calculated to be

$$\delta O = 0, \quad \delta O' = \sqrt{\frac{1}{2N}}, \tag{28}$$

with $O = A_{0x}, A_{0y}, A_{+x}, A_{+y}, A_{-x}, A_{-y}$, and $O' = S_{x,y,z}$. For sufficiently large N , we note that $\delta O' \approx 0$. We have scaled the time variable by u_0 and fixed $u_2/u_0 = -0.01$ for a ferromagnetically interacting condensate.

We note that in the self-trapped regime, the number of atoms in each well does not change appreciably. This allows us to use the mode functions with the ground-state solution of the GPE equation (6). This enhances the range of validity of two-mode approximation to that of SMA. In the Rabi regime, where the number of atoms in each well varies in time, instead of the solution to Eq. (6), we use the solution of Eq. (7). In this case, two-localized-noninteracting-spatial-mode description is only valid for a relatively small number of atoms or weak interactions. The validity of two-mode approximation in the Rabi regime is given by $N \ll r_0/a$ [4], where r_0 is the ground-state width of the effective harmonic trap potential when many body effects are discarded, and $a = (a_0 + 2a_2)/3$. For $r_0 = 10 \mu\text{m}$ and $a = 5 \text{ nm}$, the condition becomes $N \ll 2000$ [4], a value supported by the experimental parameters [1]. Strictly speaking, the two-mode approxima-

FIG. 1. Time dependence of $S_{x,y,z}$ in the self-trapping regime for $N=10^3$ atoms. Initially, the condensate is assumed to be prepared in a coherent state so that $\Psi(0)=|0, \alpha, 0\rangle|0, 0, 0\rangle$, where $|\alpha|^2=N$. Evolution is determined by Eqs. (22), without Bogoliubov back-reaction (BBR) terms. Time is scaled by u_0 , and spin variables are scaled by N . We take $u_2=-0.01u_0$ for ferromagnetic interaction, and $J=0.02N(u_0+u_2)$ for self-trapping.

tion is valid in the Rabi regime for a few hundred atoms. We shall consider $N=1000$ and $N=400$, somewhat intermediate between the quantum and mean field treatments from the past, such that even noninteracting spatial modes can be used safely within the SMA. We use $N=10^3$, as used in the experiments, where Josephson oscillations are observed, in qualitative agreement with the two-mode theory [1]. For a full quantitative agreement with the experimental findings, it is necessary to go beyond the two-mode model and SMA, by numerical solution of the nonpolynomial Schrödinger equation [1,83]. Similar numerical approaches for high-order quantum corrections to the mean field theory for spinor BEC are not known. We have examined smaller number of atoms as well, to check if there are any significant changes from $N=400$ and $N=1000$ results. Similar results are found with those reported here, apart from an essential difference in the period of oscillations due to increased or decreased atom number N .

A. Self-trapped regime

When the nonlinear interaction inside each well is stronger than the tunneling coupling, self-trapping occurs. Assuming $J/u_0=0.02N(1+u_2/u_0)$, the time evolution of the averaged spin variables are shown in Fig. 1. We observe that S_x and S_y oscillate with a $\pi/2$ phase difference while S_z undergoes partial oscillations. Geometrically, the spin vector is observed to precess about the z axis, making small oscillations along the otherwise circular path, almost parallel to the x - y plane.

There are various definitions of spin squeezing. The standard definition is based on the Heisenberg uncertainty principle. Whenever $[\Delta S_{x,y}]^2 < (1/2)|\langle \hat{S}_z \rangle|$, a spin system is said to be squeezed [80,81]. This definition, however, is problematic because it does not respect quantum correlations [49].

FIG. 2. Time dependence of the variances $[\Delta S_{x,y,z}]^2$ that is scaled by SQL of $N/4$ in the self trapping regime. Parameters are identical as in Fig. 1. Evolution is determined by Eqs. (22) and (25), without Bogoliubov back-reaction terms, and using $[\Delta O]^2 = (\delta O)^2/2$, where O corresponds to pseudospin (S) and spin-singlet (A) operators described in the text. Initial quantum noise is assumed to be a sphere in pseudo-spin space so that $\delta S_{x,y,z} = \sqrt{1/2N}$ (scaled by N). For the singlet operators we take $\delta O=0$.

According to that definition, a mere rotation of a coherent spin state (Bloch state) [82] would lead to squeezing [49,81]. A practical definition of spin squeezing that takes into account quantum correlations was given [49] as $2[\Delta S_n]^2/S < 1$. Here, $[\Delta S_n]^2$ is the uncertainty in the direction orthogonal to the mean total spin for an N spin-1/2 system. The existence of such an orthogonal direction for which the inequality is satisfied would be sufficient indication for a squeezed spin state. As the practically measurable quantities are the fluctuations (variances) of atom number and relative phase, we restrict ourselves to spin components in the x , y , and z directions [41]. In that case, we look for a perpendicular direction $\alpha=x,y,z$ to the mean spin vector for which quantum fluctuations go below the standard Heisenberg quantum limit (SQL) of $N/4$, i.e.,

$$\frac{4[\Delta S_\alpha]^2}{N} < 1. \quad (29)$$

In the case of self-trapping along the z direction, $\alpha=x$ or y .

We illustrate the dynamical behavior of quantum fluctuations for the spin components in Fig. 2. For convenience, we have scaled the quantum fluctuations by the SQL. We see that as the spin vector precesses about the z axis with its z component trapped at its maximal value, the associated error ellipsoid exhibits a similar dynamical behavior. Along the z direction, fluctuations are now trapped by performing small oscillations just above the SQL. Accordingly, there is no squeezing along the z direction. The symmetrical noise distribution of the initial coherent state, however, is deformed along the x and y directions, resulting in pseudo-spin-squeezing. Reducing of fluctuations below the SQL along the x,y directions is due to quantum correlations. Let us note

that, in the self-trapping regime, the standard Heisenberg-type spin squeezing criterion becomes equivalent to quantum correlated spin squeezing criterion, as $[S_x, S_y] = iS_z$.

As time progresses, the initial error sphere will evolve into a cigar shape, oscillating between the two extreme shapes periodically, an ellipse in the y - z plane and an ellipse in the z - x plane. Oscillations of the ellipsoid are not too rapid when compared to the case in the Rabi regime to be shown in Fig. 5. The squeezing as occurring in such a form of periodical oscillations is consistent with the rotating spin vector described above. One sees from Fig. 2 that there exist time intervals where a time-averaged squeezing parameter would be <1 . The dimension of quantum noise ellipsoid is bounded from above by 2 as can be seen from Fig. 2. They do not grow in time in contrast to the behavior of the error ellipsoid in the Rabi regime (to be shown in Fig. 5). Such a stable behavior of the noise ellipsoid may be exploited for achieving robust multiparticle entanglement or quantum correlations.

Though the squeezed spin states are found due to quantum correlations, it is necessary to use a generalized spin squeezing criterion to test whether these correlations are associated with quantum entanglement. The existence of multipartite quantum entanglement in our model system is characterized by

$$\xi_\alpha^2 = \frac{N[\Delta S_\alpha]^2}{\langle S_\beta \rangle^2 + \langle S_\gamma \rangle^2}, \quad (30)$$

where α, β, γ can take x, y, z . This particular orientation of axes is not an optimal choice for squeezing, but again this is the most practical one from an experimental point of view [41]. In general, spin squeezing as defined by minimization over all possible orthogonal axes is a sufficient condition for quantum entanglement [36]. On the other hand a self-trapping effect provides a unique opportunity to generate quantum entanglement out of the usual Heisenberg-type spin squeezing, which becomes identical to quantum correlated spin squeezing. Assuming self-trapping of a spin along the y direction at its maximal value of $N/2$ and noting that $\langle S_y \rangle^2 \gg \langle S_\beta \rangle^2$ at all times, we find $\xi_\alpha^2 \lesssim 4[\Delta S_\alpha]^2/N$. Hence, we conclude that Heisenberg-type pseudo-spin-squeezing in the directions perpendicular to the self-trapping direction implies quantum entanglement. Figures 3 and 4 illustrate this effect. The behavior of the entanglement parameter mimics the time dependence of the pseudo-spin-squeezing parameters in the x and y directions and exhibits the same stable and robust behavior.

Though it may not be as practical, it is intriguing to investigate other axial configurations for optimum quantum entanglement. Rotating the coordinate system by an azimuthal angle θ about the self-trapping axis (z axis), the pseudo-spin-squeezing parameter becomes

$$\xi_\theta^2 = N \frac{\Delta S_x^2 \cos^2 \theta + \Delta S_y^2 \sin^2 \theta + \langle S_x, S_y \rangle \sin \theta \cos \theta}{\langle S_z \rangle^2 + (\langle S_x \rangle \sin \theta - \langle S_y \rangle \cos \theta)^2}. \quad (31)$$

In the self-trapped regime, the denominator is dominated by $\langle S_z \rangle \sim N/2$. At $\theta=0$ and at $\theta=\pi/2$, Heisenberg squeezing of

FIG. 3. The same as in Fig. 2 except for the time dependence of spin squeezing (entanglement) parameters $[\xi_{x,y,z}]^2$.

pseudospin and the quantum entanglement characterized by ξ_θ^2 becomes identical. According to Fig. 2, ΔS_x and ΔS_y exhibit competing dynamical behaviors. As one grows the other decreases. Such a competition can be exploited to keep $\xi_\theta^2 < 1$ at all times. In order to illustrate this effect, the dynamics of ξ_θ^2 is shown in Fig. 4 for $\theta \in [0, \pi/2]$. When Heisenberg pseudo-spin-squeezing occurs in the x direction, $\theta=0$ ensures quantum entanglement ($\xi_\theta^2 < 1$), whereas when the Heisenberg pseudo-spin-squeezing is in the y direction, one can choose $\theta=\pi/2$. In the intermediate regions, substantial entanglement could be achieved with $\xi_\theta^2 \approx 10^{-4}$ corresponding to a θ given by

$$\theta = \frac{1}{2} \arctan\left(\frac{\langle S_x, S_y \rangle}{\Delta S_x^2 - \Delta S_y^2}\right) + \frac{\pi}{2}k, \quad (32)$$

where k is an integer. Note that, by allowing θ to be chosen from the full domain of $[0, 2\pi]$, substantial entanglement

FIG. 4. The same as in Fig. 3 except for the time dependence of the spin squeezing (entanglement) parameter $[\xi_\theta]^2$, in a coordinate system rotated by angle $\theta(t)$ about the self-trapping axis.

would be obtained at all times. However, this is not a very practical situation with regard to the measurement as θ has to be changed continuously, according to Eq. (32). On the other hand, the above scenario of exploiting the Heisenberg-type squeezing for entanglement allows for constant $\theta=0, \pi/2$ over regular finite time intervals and seems accessible experimentally.

In order to emphasize characterization of quantum entanglement in the system in terms of spin squeezing, we note that relation between more direct measures of bipartite entanglement and spin squeezing in collective spin systems have been pointed out very recently [84–86]. According to these studies, if there exists mutually orthogonal directions $\vec{k}, \vec{l}, \vec{n}$ such that the inequality

$$\langle J_{\vec{n}}^2 \rangle + \frac{N(N-2)}{4} < \sqrt{\left[\langle J_{\vec{k}}^2 \rangle + \langle J_{\vec{l}}^2 \rangle - \frac{N}{2} \right]^2 + (N-1)^2 \langle J_{\vec{n}} \rangle^2}$$

holds, then the state of the system possesses bipartite entanglement. For symmetric states, this criterion is both necessary and sufficient. In that case, it can be simplified to

$$\frac{4\langle \Delta J_{\vec{n}}^2 \rangle}{N} < 1 - \frac{4\langle J_{\vec{n}}^2 \rangle^2}{N^2}. \quad (33)$$

We see that in the case of self-trapping along the γ direction, we can find a direction α perpendicular to γ axis such that $4\langle \Delta J_{\vec{\alpha}}^2 \rangle / N < 1$. This is the same with our conclusion that Heisenberg spin squeezing criterion becomes identical to quantum correlated spin squeezing condition, which further implies quantum entanglement spin squeezing criterion $\xi_{\alpha}^2 < 1$.

B. Dynamics in the Rabi regime

When nonlinear interactions in the wells are weaker than the tunneling coupling, complete oscillations between the wells occur. As an example we take $J/u_0 = 2N(1+u_2/u_0)$. The time evolution of spin variables is shown in Fig. 5. We observe that the circular behavior of the spin vector is transferred from the x - y to the y - z plane.

From the dynamics of components of the mean spin, we see that the total mean spin is approximately confined in the yz plane. The quantum correlated pseudo-spin-squeezing criterion now reads $4[\Delta S_x]^2 / N < 1$. Variances can go below SQL in all directions, as shown in Fig. 6, due to rotation of the noise ellipsoid. Consistent with the circular behavior of the spin vector in the y - z plane, oscillations of fluctuations in the z and y directions are out of phase, while the x component of the spin vector remains small with its corresponding fluctuation oscillating around the SQL. Although the amplitude of oscillations grow in time, the fluctuations remain smaller than those along the y and z directions. The initial noise sphere will quickly turn into an ellipsoid growing in all directions making periodic oscillations into an ellipse in the x - y and x - z planes. Although the periodic shrinking of the noise ellipsoid in the y and z directions may point to pseudo-spin-squeezing, such squeezing happens too rapidly, which may hinder its practical usefulness in comparison to squeezing found in the Josephson self-trapping regime.

FIG. 5. Time dependence of scaled pseudospin components $S_{x,y,z}$ in the Rabi (oscillatory) regime with $J=2N(u_0+u_2)$. The details and other parameters are identical to those in Fig. 1.

We note that the Heisenberg spin squeezing is not identical to the quantum correlated spin squeezing in the Rabi regime. On the other hand, because the mean spin along the x direction is much smaller than the total spin S , we can write $\xi_x^2 = N[\Delta S_x]^2 / (S^2 - \langle S_x \rangle^2) \approx 4[\Delta S_x]^2 / N$. This shows the quantum correlations leading to spin squeezing are of the entangled type. It can also be compared to the criterion of bipartite entanglement given Eq. (33) with $\vec{n}=x$ direction. It leads to the same $4[\Delta S_x]^2 / N < 1$ condition. As shown in Fig. 7, quantum entanglement can develop significantly in time. Quantum entanglement can be detected by computing the entanglement parameters from the measurement outcomes

FIG. 6. The same as in Fig. 2 except for the time dependence of scaled variances $[\Delta S_{x,y,z}]^2$ in the Rabi (oscillatory) regime with $J=2N(u_0+u_2)$. The scaled variance of 1 in the graphs corresponds to SQL of $N/4$. The curves are continuous but their portions above 2 are not shown to present the squeezing regions below SQL more clearly. The time axis is given in logarithmic scale to show the early evolution more explicitly.

FIG. 7. Time dependence of the spin squeezing (entanglement) parameters in $[\xi_{x,y,z}]^2$ in the Rabi (oscillatory) regime. Details and parameters are the same as in Fig. 6. The curves are continuous, but only their portions below 2 are shown for the sake of clarity in presentation of their behavior in the entanglement region below 1.

along the y and z directions. In the x direction, quantum correlations grow more slowly in accordance with the more stable behavior of squeezing in this direction compared to the other directions. However, the oscillations are too rapid when compared to the pseudo-spin-squeezing time scale. We also observe that entanglement in the Rabi regime is more oscillatory than observed in the self-trapped situation, where entanglement is found to be more stable with a time-averaged squeezing parameter less than unity.

C. Tunneling of mode-entangled particles

Spinor BEC in double-well potential offers a rich variety of possibilities to alter its dynamics due to the additional degree of freedom in the atomic internal states. More importantly, this can be used to control quantum correlations in time. If we take an initial condition in which there is no mass tunneling so that $\langle S_z \rangle = 0$, spinor-BEC might behave differently in time from the system of a scalar or even two-mode condensate in a double-well potential. Let us consider a case where half of the atoms is in one of the wells with spin $+1$, while the other half is in the other well with spin -1 . After tunneling is initiated, these atoms would eventually get entangled due to spin mixing atomic collisions in each well. Their distributions between the wells further lead to multiparticle entanglement, which is witnessed directly by the spin squeezing parameter ξ_α^2 . Specifically, let us consider an initial coherent state such that $\Psi(0) = |\alpha, 0, 0\rangle |0, 0, \alpha\rangle$ with $\alpha = \sqrt{N}/2$. The evolution of the spin squeezing parameters are presented in Fig. 8. This figure is obtained for the case where BBR terms are kept, although the linearized treatment gives essentially the same result. Spin squeezing is found in the y direction in a given time range. Spin squeezing is possible only when u_2 is nonzero. Spin mixing allows for the growth of spin singlet pairs in time. We have found that the spin squeezing criterion is satisfied at later times as well, such as

FIG. 8. Time dependence of spin squeezing (entanglement) parameters $[\xi_{x,y,z}]^2$ in the self-trapping regime for an initial condition where $\Psi(0) = |\alpha, 0, 0\rangle |0, 0, \alpha\rangle$ with $\alpha = \sqrt{N}/2$. Time evolution is determined by the coupled equations (22) and (25) with BBR terms included. The details and parameters are the same as in Fig. 3. All the curves are continuous, but only the portions related to entanglement and spin squeezing criterion $[\xi_{x,y,z}]^2 < 1$ are shown for clarity.

at $t \sim 0.05, 0.01, 0.015$, and so on. Squeezing in the z direction is found at $t \sim 0.03$.

Alternatively, one can employ the Feshbach resonance technique to tune scattering length or change the interaction coefficients u_0 and u_2 . An intriguing situation arises when $u_2 = -u_0$. In this case, the second term disappears from the Hamiltonian (17). For an initial state of the form $|0, n_{10}, 0\rangle |0, 0, 0\rangle$, where only magnetic sublevel $m_1 = 0$ in the first well is populated, the model system describes tunneling of mode-entangled particles. Tunneling redistributes mode-entangled atoms, by the spin-mixing interaction, between the quantum wells. This leads to pseudo-spin-squeezing associated with multiparticle entanglement. For $N = 400$ atoms, we demonstrate in Fig. 9 that this effect manifests itself in the short time dynamics. We assumed that u_2 is raised in magnitude up to u_0 and took $J = 0.0198N$ (in units of u_0), the same value used for Fig. 3. Despite small tunneling strength, we still limit ourselves to small condensate case with $N = 400$, as the increase of u_2 to the values comparable to u_0 would invalidate the single-mode approximation. For such small condensates, the two-mode approximation would still work, as argued before for the Rabi regime.

D. Effect of quantum back reaction

In order to study dynamics beyond the linearized treatment, we now consider second-order dynamics by including of the Bogoliubov back-reaction (BBR) terms to the mean-field dynamics. More explicitly, we solve for the correction $\langle \delta O_j \rangle$ equation (25) coupled with the first-order moment $\langle O_j \rangle$ equation (22). The coupling of these two sets of equations results in modifications, or back reactions to the mean field evolution, in contrast to the linearized treatment. The phase space trajectories in this case are no longer restricted on the surface defined by the mean field solution. As a result, the uncertainty ellipse might become distorted, indicating spin

FIG. 9. Time dependence of spin squeezing (entanglement) parameters $[\xi_{x,y,z}]^2$ in the self-trapping regime for an initial condition where $\Psi(0)=|0,\alpha,0\rangle|0,0,0\rangle$ with $\alpha=\sqrt{N}$. Time evolution is determined by the coupled equations (22) and (25) with BBR terms included. The parameters used are $N=400$, $u_2=-u_0$, and $J=0.0198N$ (in units of u_0). The other details and parameters are the same as in Fig. 3. All the curves are continuous, but only the portions related to entanglement and spin squeezing criterion $[\xi_{x,y,z}]^2 < 1$ are shown for clarity.

squeezing as time goes by. To estimate the time range for the validity of the linearized treatment for discussing spin squeezing and associated quantum entanglement, corresponding parameters ξ_α^2 are computed using the solutions of the coupled equations (25) and (22).

The corresponding results are presented in Figs. 10 and 11, which should be compared to Figs. 7 and 3. We see that the linearized treatment is valid for a shorter time scale, ~ 0.015 (scaled by u_0) for $N=10^3$ atoms. We have taken the initial fluctuations to start from 0.02, but due to squeezing they can grow in any directions causing the linearization treatment to lose its validity. Hence, we calculated spin

FIG. 10. Time dependence of spin squeezing (entanglement) parameters $[\xi_{x,y,z}]^2$ in the self-trapping regime. The parameters and initial conditions are identical to those in Fig. 3. Time evolution is determined by the coupled equations (22) and (25) with BBR terms included.

FIG. 11. Time dependence of spin squeezing (entanglement) parameters $[\xi_{x,y,z}]^2$ in the Rabi regime. Time evolution is determined by the coupled equations (22) and (25) with BBR terms included. Identical figures are obtained when BBR terms are dropped. The effect of BBR is found to be negligible in the Rabi regime. The parameters and initial conditions are identical to those in Fig. 7, except that $N=400$ is used here to observe the effect of N on the time range. All the curves are continuous, but only the portions related to entanglement and the spin squeezing criterion $[\xi_{x,y,z}]^2 < 1$ are shown for clarity.

squeezing parameter directly in the presence of BBR rather than examining only first-order trajectories in the phase space. Spin squeezing is found to occur at much earlier times than the quantum break time of 0.015. This is perhaps not surprising as the original Hamiltonian is analogous to scalar or two-mode condensates in a double-well potential. In contrast to a spin-1 condensate in a double-well potential, such systems are exactly solvable and their entanglement and spin squeezing properties are well known [41]. As the evolution of the system, including the second-order correction, is closer to the exact dynamics, persistence of spin squeezing in this case might be expected. Furthermore, in the self-trapped regime, quantum fluctuations are more stable and do not grow too much from their initial values in the time ranges considered here.

In the Rabi regime for $N=400$, the results are given in Fig. 11, which should be compared to that of Fig. 7. In this case, the period of oscillations is larger, due to a decrease of N . The effect of BBR is found to be negligible in the Rabi regime. This might be expected as BBR terms contribute through interaction strengths u_0, u_2 , which are much less than J in the Rabi regime. It is also known that improvements of the mean field theory is usually desirable in the self-trapping regime, rather than in the Rabi regime [77]. Macroscopic self-trapping in the pseudospin space occur at a dynamically unstable point about which unitary mean field evolution breaks down on a time scale (“quantum break time”) logarithmic in N [77].

VI. CONCLUSION

We studied quantum correlations of a spin-1 condensate inside a double-well potential. We suggested that mode en-

tanglement generated by spin-singlet interactions within each well can be distributed by the tunneling coupling into both wells. This contributes to multipartite entanglement among atoms inside different wells. This indirect route to multipartite entanglement is in addition to the usual spin squeezing route obtained via a single-axis or two-axis twisting interaction for atoms located inside different wells. Besides the substantial entanglement developed in the model system, we also realized that the self-trapping effect in the lower end of the Josephson regime can be exploited for generating pseudo-spin-squeezing and making the entanglement robust. By adjusting suitable system parameters through Feshbach resonances or changing the tunneling barrier strength, our model system can be easily controlled to fall into the self-trapping regime of nonlinear Josephson dynamics. In this regime, self-trapping of a large spin component leads to squeezing in the associated transverse components, which, in turn, implies entanglement; although, in general, spin squeezing is not a necessary condition for quantum entanglement.

Central to our understanding is the fact that the nonlinear nature of our model combined with quantum correlation generation and distribution between the double wells allows for quantum entanglement via pseudo-spin-squeezing. One can also employ Feshbach resonances to turn the mode entanglement and particle entanglement interactions on and off. An

important feature for a spatial double-well system, as in our model, is the possibility of individually addressing each well with optical pulses. Using polarized laser beams, one can access all hyperfine spin components $|0\rangle$ and $|\pm\rangle$. Normally, the mode entanglement criterion is an approximate entanglement measure for $F=1$ atoms because of the presence of internal state $|m_F=0\rangle$. The contribution of mode entanglement to particle entanglement makes it possible to use pseudo-spin-squeezing criterion *per se* to characterize useful entanglement in a spin-1 condensate in a double-well potential. Finally, we note that such quantum entanglement can be directly measured via the Ramsey spectroscopy.

Our studies reveal that a condensate of spin-1 atoms inside a double-well potential, with its flexibility for control and richness in entanglement types, presents a useful system for studying quantum coherent phenomena and potentially for implementing rudimentary quantum information processing.

ACKNOWLEDGMENTS

O.E.M. acknowledges the support from a TÜBA/GEBİP award. He also thanks ITP of Chinese Academy of Sciences in Beijing and Center for Advanced Studies at Tsinghua University for hospitality. L.Y. is supported by US NSF and NASA.

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