Dynamics of quantum phases in a spinor condensate

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We discuss the quantum phases and their diffusion dynamics in a spinor-1 atomic Bose-Einstein condensate. For ferromagnetic interactions, we obtain the exact ground-state distribution of the phases associated with the total atom number (N), the total magnetization ($\mathcal{M}$), and the alignment (or hypercharge) (Y) of the system. The mean-field ground state is stable against fluctuations of atom numbers in each of the spin components, and the phases associated with the order parameter for each spin components diffuse while dynamically recovering the two broken continuous symmetries [U(1) and SO(2)] when N and $\mathcal{M}$ are conserved, as in current experiments. We discuss the implications to the quantum dynamics due to an external (homogeneous) magnetic field. We also comment on the case of a spinor-1 condensate with antiferromagnetic interactions.

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I. INTRODUCTION

Since the observation of Bose-Einstein condensation in trapped dilute alkali atoms [1–3], the coherence properties of a condensate have become the focus of many theoretical studies [4–9]. As it was initially pointed out in Refs. [4,5], the number fluctuations associated with the assumption of a coherent-state macroscopic condensate wave function lead to the “diffusion” (or spreading) of the initial phase. For a scalar, one-component atomic condensate, this diffusion, the attempt to restore the U(1) symmetry of the underlying interacting system, can be formulated in terms of the dynamics of a zero mode, or the Goldstone mode [4]. More physically meaningful discussions, in terms of the relative phase of the two condensates, were carried out soon afterwards [6–9]. Starting with the remarkable direct observation of the first-order coherence in an interference experiment [10], direct relationship between the number fluctuation and quantum phase dynamics was first observed with a condensate in a periodic potential [11], and more recently, in the remarkable Mott insulating state [12,13].

The emergence of spinor condensates [14,15] (of atoms with hyperfine quantum number $F=1$) has created new opportunities to understand quantum coherence and the associated phase dynamics within a three-component condensate [16,17]. In an earlier paper [18], we have presented important physical insight into quantum phase diffusions of a spinor condensate. We have investigated the coupled zero mode dynamics as a consequence of the usual mean-field treatment that calls for the breaking of two continuous symmetries: a U(1) gauge transformation $e^{i\theta}$ and SO(3) spin rotations $\mathcal{U}(\alpha, \beta, \tau) = e^{-iF_z\alpha} e^{-iF_y\beta} e^{-iF_z\tau}$ (in the absence of an external magnetic field) [19].

This paper provides a detailed investigation of the quantum phase dynamics for a spinor-1 condensate. As before, we will focus on the case of ferromagnetic interactions, for which the validity of the single spatial mode approximation proves to be a convenient starting point for the emergence of transparent physical illustrations of the zero mode dynamics [20]. The fluctuations of atom numbers in each of the spin states will be shown to be connected directly, in this case, to

the fluctuations of the pointing direction of the macroscopic spin (or magnetic dipole moment) of the condensate. This paper is organized as follows. In Sec. II, after introducing the Hamiltonian for a spinor-1 condensate, we explicitly work out the Heisenberg equations for the atomic field operators. Following the standard Bogoliubov approach [21], we obtain the coupled Gross-Pitaevskii equations for the mean field of the condensate and the coupled Bogoliubov-de Gennes equations for the quantum fluctuations. We then generalize the Hermitian operators for the condensate number and phase fluctuations (of the Goldstone mode), as introduced earlier [4], to each of the spin components, and derive their dynamic equations under the rotating wave-function approximation. In Sec. III, we perform a detailed study of the stationary phase diffusion of a spinor-1 condensate within the single mode approximation (SMA) valid exactly for ferromagnetic interactions. Section IV addresses the influence of an external magnetic field and in Sec. V we discuss the same phase dynamics for a condensate with antiferromagnetic interactions. We conclude in Sec. VI.

II. FORMULATION

We consider a system of N spin-1 bosonic atoms interacting via only s -wave scattering [19,22–24]. In the second-quantized form, the Hamiltonian of our system becomes [19,22]

$$\begin{aligned}
 H = & \sum_i \int d\vec{r} \psi_i^\dagger(\vec{r}) \left[-\frac{\hbar^2 \nabla^2}{2M} + V_{\text{ext}}(\vec{r}) \right] \psi_i(\vec{r}) \\
 & + \frac{c_0}{2} \sum_{i,j} \int d\vec{r} \psi_i^\dagger(\vec{r}) \psi_j^\dagger(\vec{r}) \psi_i(\vec{r}) \psi_j(\vec{r}) \\
 & + \frac{c_2}{2} \sum_{i,j,k,l} \int d\vec{r} \psi_i^\dagger(\vec{r}) \psi_j^\dagger(\vec{r}) \vec{F}_{ik} \cdot \vec{F}_{jl} \psi_l(\vec{r}) \psi_k(\vec{r}), \quad (1)
 \end{aligned}$$

where $\psi_j(\vec{r})$ ($j = +, 0, -$) denotes the annihilation operator for the j th component of a spinor-1 field. The external trapping potential $V_{\text{ext}}(\vec{r})$ is assumed to be spin independent, as in a far off-resonant optical dipole force trap so that atomic

spin degrees of freedom are completely accessible. The pair interaction coefficients are $c_0 = 4\pi\hbar^2(a_0 + 2a_2)/3M$ and $c_2 = 4\pi\hbar^2(a_2 - a_0)/3M$, with a_0 (a_2) being the s -wave scattering length for two spin-1 atoms in the combined symmetric channel of total spin 0 (2), and M is the mass of atom. $\vec{F}$ is spin-1 matrix representation. The interacting terms can be regrouped to show that they include, respectively, self-scattering, cross scattering, and the spin relaxation [24].

From Hamiltonian (1), one can easily derive the Heisenberg equations for field operators

$$\begin{aligned} i\hbar\dot{\psi}_+ &= \left[-\frac{\hbar^2\nabla^2}{2M} + V_{\text{ext}}(\vec{r}) \right] \psi_+ + c_2\psi_-^\dagger\psi_0^2 + c_0(\psi_+^\dagger\psi_+ \\ &\quad + \psi_0^\dagger\psi_0 + \psi_-^\dagger\psi_-)\psi_+ + c_2(\psi_+^\dagger\psi_+ + \psi_0^\dagger\psi_0 \\ &\quad - \psi_-^\dagger\psi_-)\psi_+, \\ i\hbar\dot{\psi}_0 &= \left[-\frac{\hbar^2\nabla^2}{2M} + V_{\text{ext}}(\vec{r}) \right] \psi_0 + 2c_2\psi_0^\dagger\psi_+\psi_- + c_0(\psi_+^\dagger\psi_+ \\ &\quad + \psi_0^\dagger\psi_0 + \psi_-^\dagger\psi_-)\psi_0 + c_2(\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-)\psi_0, \\ i\hbar\dot{\psi}_- &= \left[-\frac{\hbar^2\nabla^2}{2M} + V_{\text{ext}}(\vec{r}) \right] \psi_- + c_2\psi_+^\dagger\psi_0^2 + c_0(\psi_+^\dagger\psi_+ \\ &\quad + \psi_0^\dagger\psi_0 + \psi_-^\dagger\psi_-)\psi_- + c_2(\psi_-^\dagger\psi_- + \psi_0^\dagger\psi_0 \\ &\quad - \psi_+^\dagger\psi_+)\psi_-. \end{aligned} \quad (2)$$

Following the Hartree mean-field theory, we assume that there are three “large” condensate components ϕ_j around which we can study the small quantum fluctuations (off-condensate excitations) via the Bogoliubov approximation

$$\psi_j(\vec{r}) = \sqrt{N}\phi_j(\vec{r}) + \delta\psi_j(\vec{r}), \quad (3)$$

with N being the total number of atoms. We note that such an approach can become questionable in a multicomponent system where a certain component becomes negligibly small and its quantum nature becomes important. For ferromagnetic interactions, however, our previous studies [20] show that the ground-state wave functions can always be expressed as $\phi_j \propto \phi$ and hence the Bogoliubov prescription can always be applied. By substituting Eq. (3) into Eq. (2), we find, in the zeroth order, the coupled Gross-Pitaevskii equations (GPEs)

$$\begin{aligned} i\hbar\dot{\phi}_+ &= [\mathcal{L} + c_0\rho + c_2(\rho_+ + \rho_0 - \rho_-)]\phi_+ + c_2N\phi_-^*\phi_0^2, \\ i\hbar\dot{\phi}_0 &= [\mathcal{L} + c_0\rho + c_2(\rho_+ + \rho_-)]\phi_0 + 2c_2N\phi_0^*\phi_+\phi_-, \\ i\hbar\dot{\phi}_- &= [\mathcal{L} + c_0\rho + c_2(\rho_- + \rho_0 - \rho_+)]\phi_- + c_2N\phi_+^*\phi_0^2, \end{aligned} \quad (4)$$

with $\mathcal{L} = -\hbar^2\nabla^2/2M + V_{\text{ext}}(\vec{r})$, j th component condensate density $\rho_j(\vec{r}) = N|\phi_j(\vec{r})|^2$, and total condensate density $\rho(\vec{r}) = \rho_+(\vec{r}) + \rho_0(\vec{r}) + \rho_-(\vec{r})$. $\int d\vec{r}\rho_j(\vec{r}) = N_j$ is the number of condensed atoms in the j th component. In the first order, the quantum fluctuations obey the following set of Bogoliubov–de Gennes equations (BdGEs):

$$\begin{aligned} i\hbar\frac{\partial\delta\psi_+}{\partial t} &= [\mathcal{L} + c_0(\rho + \rho_+) + c_2(2\rho_+ + \rho_0 - \rho_-)]\delta\psi_+ \\ &\quad + N(c_+\phi_0^*\phi_+ + 2c_2\phi_-^*\phi_0)\delta\psi_0 \\ &\quad + c_-N\phi_-^*\phi_+\delta\psi_- + c_+N\phi_+^2\delta\psi_+^\dagger \\ &\quad + c_+N\phi_+\phi_0\delta\psi_0^\dagger + N(c_-\phi_+\phi_- + c_2\phi_0^2)\delta\psi_-^\dagger, \\ i\hbar\frac{\partial\delta\psi_0}{\partial t} &= [\mathcal{L} + c_0(\rho + \rho_0) + c_2(\rho_+ + \rho_-)]\delta\psi_0 + N(c_+\phi_+^*\phi_0 \\ &\quad + 2c_2\phi_0^*\phi_-)\delta\psi_+ + N(c_+\phi_-^*\phi_0 \\ &\quad + 2c_2\phi_0^*\phi_+)\delta\psi_- + c_+N\phi_+\phi_0\delta\psi_+^\dagger \\ &\quad + c_+N\phi_0\phi_-\delta\psi_-^\dagger + N(c_0\phi_0^2 + 2c_2\phi_+\phi_-)\delta\psi_0^\dagger, \\ i\hbar\frac{\partial\delta\psi_-}{\partial t} &= [\mathcal{L} + c_0(\rho + \rho_-) + c_2(2\rho_- + \rho_0 - \rho_+)]\delta\psi_- \\ &\quad + N(c_+\phi_0^*\phi_- + 2c_2\phi_+^*\phi_0)\delta\psi_0 \\ &\quad + c_-N\phi_+^*\phi_-\delta\psi_+ + c_+N\phi_-^2\delta\psi_-^\dagger \\ &\quad + c_+N\phi_-\phi_0\delta\psi_0^\dagger + N(c_-\phi_+\phi_- + c_2\phi_0^2)\delta\psi_+^\dagger, \end{aligned} \quad (5)$$

where we have defined $c_\pm \equiv c_0 \pm c_2$.

To study the quantum phase dynamics, we introduce number fluctuation operators [4]

$$P_j = \int d\vec{r} [\phi_j^*(\vec{r})\delta\psi_j(\vec{r}) + \phi_j(\vec{r})\delta\psi_j^\dagger(\vec{r})]. \quad (6)$$

Using GPEs (4) and BdGEs (5) above, we find the following evolution equations for these operators:

$$\begin{aligned} i\hbar\frac{\partial P_+}{\partial t} &= -\frac{i\hbar}{2}\frac{\partial P_0}{\partial t} = i\hbar\frac{\partial P_-}{\partial t} = c_2N \int d\vec{r} (\phi_-^*\phi_0^2\delta\psi_+^\dagger \\ &\quad + \phi_+^*\phi_0^2\delta\psi_-^\dagger + 2\phi_+^*\phi_-^*\phi_0\delta\psi_0 - \text{H.c.}). \end{aligned} \quad (7)$$

Thus, both $P_{\text{tot}} = P_+ + P_0 + P_-$ and $P_+ - P_-$ are constants of motion, which is, in fact, obvious since Hamiltonian (1) conserves the total number of atoms and the total magnetization.

We define conjugate phase operator $Q_{\text{tot}} = Q_+ + Q_0 + Q_-$ according to

$$Q_j = i\hbar \int d\vec{r} [\theta_j^*(\vec{r}) \delta\psi_j(\vec{r}) - \theta_j(\vec{r}) \delta\psi_j^*(\vec{r})], \quad (8)$$

where $\theta_j(\vec{r})$ are the phase functions to be determined [4]. Canonical quantization condition $[Q_{\text{tot}}, P_{\text{tot}}] = i\hbar$ is satisfied if constraint $J_+ + J_0 + J_- \equiv 1$ is enforced with $J_j \equiv \int d\vec{r} [\theta_j^*(\vec{r}) \phi_j(\vec{r}) + \theta_j(\vec{r}) \phi_j^*(\vec{r})]$. From the defining equation for Q_{tot} ,

$$\frac{dQ_{\text{tot}}}{dt} = N\tilde{u}P_{\text{tot}}, \quad (9)$$

the Goldstone mode inertial parameter $\tilde{u}$ can be determined. We obtain the dynamic equations for the phase functions as

$$\begin{aligned} i\hbar \dot{\theta}_+ = & [\mathcal{L} + c_0(\rho + \rho_+) + c_2(2\rho_+ + \rho_0 - \rho_-)]\theta_+ \\ & + N(c_+ \phi_0^* \phi_+ + 2c_2 \phi_-^* \phi_0) \theta_0 + c_- N \phi_-^* \phi_+ \theta_- \\ & + c_+ N \phi_+^2 \theta_+^* + c_+ N \phi_+ \phi_0 \theta_0^* + N(c_- \phi_+ \phi_- \\ & + c_2 \phi_0^2) \theta_-^* - N\tilde{u} \phi_+, \end{aligned}$$

$$\begin{aligned} i\hbar \dot{\theta}_0 = & [\mathcal{L} + c_0(\rho + \rho_0) + c_2(\rho_+ + \rho_-)]\theta_0 + N(c_+ \phi_+^* \phi_0 \\ & + 2c_2 \phi_0^* \phi_-) \theta_+ + N(c_+ \phi_-^* \phi_0 + 2c_2 \phi_0^* \phi_+) \theta_- \\ & + c_+ N \phi_+ \phi_0 \theta_+^* + c_+ N \phi_0 \phi_- \theta_-^* + N(c_0 \phi_0^2 \\ & + 2c_2 \phi_+ \phi_-) \theta_0^* - N\tilde{u} \phi_0, \end{aligned}$$

$$\begin{aligned} i\hbar \dot{\theta}_- = & [\mathcal{L} + c_0(\rho + \rho_-) + c_2(2\rho_- + \rho_0 - \rho_+)]\theta_- \\ & + N(c_+ \phi_0^* \phi_- + 2c_2 \phi_+^* \phi_0) \theta_0 + c_- N \phi_+^* \phi_- \theta_+ \\ & + c_+ N \phi_-^2 \theta_-^* + c_+ N \phi_- \phi_0 \theta_0^* + N(c_- \phi_+ \phi_- \\ & + c_2 \phi_0^2) \theta_+^* - N\tilde{u} \phi_-. \end{aligned} \quad (10)$$

Combining Eqs. (5) and (10), the dynamic equations of Q_j can be easily obtained to be

$$\begin{aligned} \dot{Q}_+ = & N\tilde{u}P_+ + N \int d\vec{r} \{ -[(c_+ \phi_+^* \phi_0 + 2c_2 \phi_0^* \phi_-) \theta_0^* + c_+ \phi_+^* \phi_0^* \theta_0 + (c_- \phi_+^* \phi_-^* + c_2 \phi_0^{*2}) \theta_- + c_- \phi_+^* \phi_- \theta_-^*] \delta\psi_+ \\ & + (c_+ \phi_0^* \phi_+ \theta_+^* + c_+ \phi_+^* \phi_0^* \theta_+ + 2c_2 \phi_-^* \phi_0 \theta_+^*) \delta\psi_0 + (c_- \phi_+^* \phi_-^* \theta_+ + c_- \phi_-^* \phi_+ \theta_+^* + c_2 \phi_0^{*2} \theta_+) \delta\psi_- + \text{H.c.} \}, \end{aligned}$$

$$\begin{aligned} \dot{Q}_0 = & N\tilde{u}P_0 + N \int d\vec{r} \{ -[(c_+ \phi_0^* \phi_+ + 2c_2 \phi_-^* \phi_0) \theta_+^* + c_+ \phi_+^* \phi_0^* \theta_+ + (c_+ \phi_0^* \phi_- + 2c_2 \phi_+^* \phi_0) \theta_-^* + c_+ \phi_0^* \phi_-^* \theta_-] \delta\psi_0 \\ & + (c_+ \phi_+^* \phi_0 \theta_0^* + c_+ \phi_+^* \phi_0^* \theta_0 + 2c_2 \phi_0^* \phi_- \theta_0^*) \delta\psi_+ + (c_+ \phi_-^* \phi_0 \theta_0^* + c_+ \phi_0^* \phi_-^* \theta_0 + 2c_2 \phi_0^* \phi_+ \theta_0^*) \delta\psi_- + \text{H.c.} \}, \end{aligned}$$

$$\begin{aligned} \dot{Q}_- = & N\tilde{u}P_- + N \int d\vec{r} \{ -[(c_+ \phi_-^* \phi_0 + 2c_2 \phi_0^* \phi_+) \theta_0^* + c_+ \phi_-^* \phi_0^* \theta_0 + (c_- \phi_+^* \phi_-^* + c_2 \phi_0^{*2}) \theta_+ + c_- \phi_-^* \phi_+ \theta_+^*] \delta\psi_- \\ & + (c_+ \phi_0^* \phi_- \theta_-^* + c_+ \phi_-^* \phi_0^* \theta_- + 2c_2 \phi_+^* \phi_0 \theta_-^*) \delta\psi_0 + (c_- \phi_+^* \phi_-^* \theta_- + c_- \phi_+^* \phi_- \theta_-^* + c_2 \phi_0^{*2} \theta_-) \delta\psi_+ + \text{H.c.} \}. \end{aligned} \quad (11)$$

We note that $[Q_j, P_k] = i\hbar \delta_{jk} J_j$, $[\delta\psi_j(\vec{r}), P_k] = \delta_{jk} \phi_j(\vec{r})$, and $[\delta\psi_j(\vec{r}), Q_k] = -i\hbar \delta_{jk} \theta_j(\vec{r})$. This prompts us to make the rotating wave approximation (RWA) [7,9] by assuming that $\delta\psi_j(\vec{r}) = \sum_j a_j(\vec{r}) P_j + \sum_j b_j(\vec{r}) Q_j$, which leads to $a_j(\vec{r})$

$= \theta_j(\vec{r})/J_j$ and $b_j(\vec{r}) = \phi_j(\vec{r})/i\hbar J_j$. Therefore, under the RWA, we obtain $\delta\psi_j(\vec{r}) = \theta_j(\vec{r}) P_j/J_j + \phi_j(\vec{r}) Q_j/i\hbar J_j$. With this result, the complete dynamic equations for the number and phase fluctuation operators become

$$\begin{aligned}
 \dot{P}_+ &= -\frac{\dot{P}_0}{2} \\
 &= \dot{P}_- = \frac{2c_2N}{\hbar} \left[\frac{(\phi_- \theta_+ | \phi_0^2)''}{J_+} P_+ - 2 \frac{(\phi_0 \theta_0 | \phi_+ \phi_-)''}{J_0} P_0 + \frac{(\phi_+ \theta_- | \phi_0^2)''}{J_-} P_- \right] + \frac{2c_2N}{\hbar^2} (\phi_+ \phi_- | \phi_0^2)' \left(\frac{Q_+}{J_+} - \frac{2Q_0}{J_0} + \frac{Q_-}{J_-} \right), \\
 \dot{Q}_+ &= N\tilde{u}P_+ - \frac{2N}{J_+} [2c_- I_{-+} + 2c_+ I_{+0} + 2c_2(\phi_0 \theta_0 | \phi_- \theta_+)' + c_2(\phi_0^2 | \theta_- \theta_+)] P_+ + \frac{4N}{J_0} [c_+ I_{+0} + c_2(\phi_0 \theta_0 | \phi_- \theta_+)] P_0 \\
 &\quad + \frac{2N}{J_-} [2c_- I_{-+} + c_2(\phi_0^2 | \theta_+ \theta_-)] P_- + \frac{2c_2N}{\hbar J_+} [2(\phi_+ \phi_- | \phi_0 \theta_0)'' + (\phi_+ \theta_- | \phi_0^2)'] Q_+ + \frac{4c_2N}{\hbar J_0} (\phi_- \theta_+ | \phi_0^2)'' Q_0 \\
 &\quad + \frac{2c_2N}{\hbar J_-} (\phi_0^2 | \theta_+ \phi_-)'' Q_-, \\
 \dot{Q}_0 &= N\tilde{u}P_0 - \frac{4N}{J_0} [c_+ I_{+0} + c_+ I_{-0} + c_2(\phi_- \theta_+ | \phi_0 \theta_0)' + c_2(\phi_+ \theta_- | \phi_0 \theta_0)] P_0 + \frac{4N}{J_+} [c_+ I_{+0} + c_2(\phi_0 \theta_0 | \phi_- \theta_+)] P_+ \\
 &\quad + \frac{4N}{J_-} [c_+ I_{-0} + c_2(\phi_0 \theta_0 | \phi_+ \theta_-)] P_- + \frac{4c_2N}{\hbar J_+} (\phi_0 \theta_0 | \phi_+ \phi_-)'' Q_+ + \frac{4c_2N}{\hbar J_0} [(\phi_0^2 | \phi_- \theta_+)'' + (\phi_0^2 | \phi_+ \theta_-)'] Q_0 \\
 &\quad + \frac{4c_2N}{\hbar J_-} (\phi_0 \theta_0 | \phi_+ \phi_-)'' Q_-, \\
 \dot{Q}_- &= N\tilde{u}P_- - \frac{2N}{J_-} [2c_- I_{-+} + 2c_+ I_{-0} + 2c_2(\phi_0 \theta_0 | \phi_+ \theta_-)' + c_2(\phi_0^2 | \theta_+ \theta_-)] P_- + \frac{4N}{J_0} [c_+ I_{-0} + c_2(\phi_0 \theta_0 | \phi_+ \theta_-)] P_0 \\
 &\quad + \frac{2N}{J_+} [2c_- I_{-+} + c_2(\phi_0^2 | \theta_- \theta_+)] P_+ + \frac{2c_2N}{\hbar J_+} (\phi_0^2 | \theta_- \phi_+)'' Q_+ + \frac{4c_2N}{\hbar J_0} (\phi_+ \theta_- | \phi_0^2)'' Q_0 + \frac{2c_2N}{\hbar J_-} [2(\phi_- \phi_+ | \phi_0 \theta_0)'' \\
 &\quad + (\phi_- \theta_+ | \phi_0^2)'] Q_-, \tag{12}
 \end{aligned}$$

where we have introduced the following shorthand notation: $(fg|hy) = \int d\vec{r} f^*(\vec{r}) g^*(\vec{r}) h(\vec{r}) y(\vec{r})$, $(\cdot)' = \text{Re}(\cdot)$, $(\cdot)'' = \text{Im}(\cdot)$, and $I_{ml} = \int d\vec{r} (\phi_m \theta_m^*)' (\phi_l \theta_l^*)'$. Equation (12) thus completely determines the zero mode condensate fluctuations. Before finding the corresponding compact zero mode Hamiltonian in quadratic forms of P_j and Q_j , we will first attempt to simplify Eq. (12), in the following section, using general properties of $\phi_j(\vec{r})$ and $\theta_j(\vec{r})$ for a ferromagnetic spinor-1 condensate [20].

III. PHASE DIFFUSIONS AND THE ZERO MODES

In general, the ground-state wave functions of a spinor condensate can be expressed as

$$\phi_j(\vec{r}, t) = |\phi_j(\vec{r})| e^{-i\mu t/\hbar + i\alpha_j}, \tag{13}$$

with a common chemical potential μ for all spin components when the external magnetic field is negligible. Based on the minimization for the total energy Eq. (1) within the mean-field theory, it was shown in Refs. [25,26] that there exists an important relation among α_j , given by $\alpha_+ + \alpha_- - 2\alpha_0 = 0$. We further proved in Ref. [20] that the ground-state wave

function for each of the spin component takes the same spatial shape for (ferromagnetic interactions), i.e.,

$$\phi_j(\vec{r}) = \sqrt{n_j} \phi(\vec{r}) e^{i\alpha_j}, \tag{14}$$

with the real-valued mode function $\phi(\vec{r})$ (normalized to unity) governed by an equivalent scalar condensate GPE

$$[\mathcal{L} + c_+ N \phi^2(\vec{r})] \phi(\vec{r}) = \mu \phi(\vec{r}), \tag{15}$$

of a scattering length a_2 (note, $c_+ \propto a_2$). Atoms, thus, only collide in the symmetric total spin $F=2$ channel in a ferromagnetic state in order to maintain their individual spins parallel. $n_j = N_j/N$ is the ratio of the number of atoms in the i th spin component to the total number of atoms. For any given magnetization $\mathcal{M}$, n_j is given explicitly by $n_{\pm} = (1 \pm m)^2/4$ and $n_0 = (1 - m^2)/2$ with $m = \mathcal{M}/N$ [20,24].

Similarly, we look for phase functions of the form

$$\theta_j(\vec{r}, t) = |\theta_j(\vec{r})| e^{-i\mu_j t/\hbar + i\alpha_j}. \tag{16}$$

Substituting Eq. (16) into Eq. (10), we find that $\theta_j(\vec{r}, t)$ also evolve in time as $e^{-i\mu t/\hbar}$. To focus on the steady-state quantum fluctuation properties, we may, therefore, neglect the

FIG. 1. The N dependence of $\tilde{u}$ (in units of $\hbar\omega_{\perp}$) for a ^{87}Rb condensate in a cylindrically symmetric trap with $\omega_x = \omega_y = \omega_{\perp} = (2\pi)100$ (Hz) and $\omega_z = \lambda\omega_{\perp}$. We have used $a_0 = 101.8a_B$ and $a_2 = 100.4a_B$ (a_B is Bohr radius) [27].

time-dependent part in $\phi_j(\vec{r}, t)$ and $\theta_j(\vec{r}, t)$ from now on. The steady-state equations for phase functions now become

$$\begin{aligned} & [\mathcal{L} - \mu + c_0(\rho + 2\rho_+) + c_2(3\rho_+ + \rho_0 - \rho_-)]\xi_+ + 2N(c_+ \phi_+ \\ & \quad + c_2 \phi_-)\phi_0 \xi_0 + N(2c_- \phi_+ \phi_- + c_2 \phi_0^2)\xi_- \\ & = N\tilde{u}\phi_+, \\ & [\mathcal{L} - \mu + c_0(\rho + 2\rho_0) + c_2(\rho_+ + 2N\phi_+ \phi_- + \rho_-)]\xi_0 \\ & \quad + 2N(c_+ \phi_+ + c_2 \phi_-)\phi_0 \xi_+ + 2N(c_+ \phi_- + c_2 \phi_+)\phi_0 \xi_- \\ & = N\tilde{u}\phi_0, \\ & [\mathcal{L} - \mu + c_0(\rho + 2\rho_-) + c_2(3\rho_- + \rho_0 - \rho_+)]\xi_- + 2N(c_+ \phi_- \\ & \quad + c_2 \phi_+)\phi_0 \xi_0 + N(2c_- \phi_+ \phi_- + c_2 \phi_0^2)\xi_+ \\ & = N\tilde{u}\phi_-, \end{aligned} \quad (17)$$

where $\xi_j(\vec{r}) = \theta_j(\vec{r})e^{-i\alpha_j}$ are now real-valued functions. One can easily check that similar to the single mode approximation for $\phi_j(\vec{r})$, $\xi_j(\vec{r})$ can also be expressed as $\xi_j(\vec{r}) = n_j^{1/2}\theta(\vec{r})$ with $\theta(\vec{r})$ governed by the following equation:

$$[\mathcal{L} - \mu + 3c_+ N\phi^2(\vec{r})]\theta(\vec{r}) = N\tilde{u}\phi(\vec{r}), \quad (18)$$

and $\tilde{u}$ determined through normalization constraint $\int d\vec{r}\phi^2(\vec{r})\theta(\vec{r}) = 1/2$. Figure 1 shows the selected results for the N dependence of $\tilde{u}$. Therefore, for ferromagnetic interactions, the phase functions can be generally expressed as

$$\theta_j(\vec{r}) = \sqrt{n_j}\theta(\vec{r})e^{i\alpha_j}. \quad (19)$$

We note that Eq. (18) is, in fact, identical to the equation satisfied by the phase function of a scalar condensate [4]; a point easily understood for ferromagnetic interactions when all spins align along the same direction. If we take this direction as the quantization axis, then the spinor condensate is essentially a scalar condensate since its total magnetization $\mathcal{M}$ is conserved. All overlap integrals in Eq. (12) are presented in Appendix A. They are all real quantities with the choice of phase parameters.

We now attempt to express the zero mode dynamics in terms of its associated Hamiltonian. Since P_j and Q_j are not

FIG. 2. The N dependence of ε (in units of $\hbar\omega_{\perp}$) for the same parameters as in Fig. 1.

canonically conjugated variables, we define $p'_j \equiv P_j$ and $q'_j \equiv Q_j/J_j$ such that they become canonical conjugates. The zero mode Hamiltonian that governs the dynamics of Eq. (12) is

$$\begin{aligned} \frac{H_{\text{zero}}}{N} = & \left(\frac{\tilde{u}}{2n_+ \mathcal{O}_{\phi\theta}} - 2c_+ \frac{1-n_+}{n_+} - 2c_2 \frac{n_-}{n_+} \right) p_+'^2 \mathcal{O}_{\phi\theta} \\ & + \left(\frac{\tilde{u}}{2n_0 \mathcal{O}_{\phi\theta}} - 2c_+ \frac{1-n_0}{n_0} - 2c_2 \right) p_0'^2 \mathcal{O}_{\phi\theta} \\ & + \left(\frac{\tilde{u}}{2n_- \mathcal{O}_{\phi\theta}} - 2c_+ \frac{1-n_-}{n_-} - 2c_2 \frac{n_+}{n_-} \right) p_-'^2 \mathcal{O}_{\phi\theta} \\ & + 4 \left[c_+ + c_2 \left(\frac{n_-}{n_+} \right)^{1/2} \right] p_+' p_0' \mathcal{O}_{\phi\theta} \\ & + 4 \left[c_+ + c_2 \left(\frac{n_+}{n_-} \right)^{1/2} \right] p_-' p_0' \mathcal{O}_{\phi\theta} + 4c_0 p_+' p_-' \mathcal{O}_{\phi\theta} \\ & - \frac{c_2 n_0^2 \mathcal{O}_{\phi\phi}}{2\hbar^2} (q_+' - 2q_0' + q_-')^2 \\ = & \mathbf{p}'^T \mathcal{A} \mathbf{p}' + \mathbf{q}'^T \mathcal{B} \mathbf{q}', \end{aligned} \quad (20)$$

where $\mathbf{p}'^T = (p_+', p_0', p_-')$, $\mathbf{q}'^T = (q_+', q_0', q_-')$, and $\mathcal{A}$ and $\mathcal{B}$ are two Hermitian matrices whose elements are easily identified from the above equation. We also present, in Appendix B, the zero mode Hamiltonian in the form of overlap integrals. To simplify notations, we have also defined $\mathcal{O}_{\phi\theta} \equiv \int d\vec{r}\phi^2(\vec{r})\theta^2(\vec{r})$ and $\mathcal{O}_{\phi\phi} \equiv \int d\vec{r}\phi^4(\vec{r})$. We find that matrix $\mathcal{B}$ is positive definite since $c_2 < 0$ for ferromagnetic spinor condensates. To show the positive definiteness of $\mathcal{A}$, we decompose $\tilde{u}$ as

$$\tilde{u} = \varepsilon + 4c_+ \mathcal{O}_{\phi\theta}, \quad (21)$$

where

$$\varepsilon = \frac{2}{N} \int d\vec{r}\theta(\vec{r})[\mathcal{L} - \mu + c_+ N\phi^2(\vec{r})]\theta(\vec{r}). \quad (22)$$

We note that $\varepsilon \equiv 0$ for noninteracting atoms [$\phi(\vec{r}) = \theta(\vec{r})$]. However, it is nonzero in general and we can prove that $\varepsilon > 0$ since $\theta(\vec{r})$ is the solution of Eq. (18) (see Fig. 2 for the N dependence of ε). By using Eq. (21), we then obtain

$$\mathbf{p}'^T \mathcal{A} \mathbf{p}' = \frac{\varepsilon}{2} \left(\frac{p_+'^2}{n_+} + \frac{p_0'^2}{n_0} + \frac{p_-'^2}{n_-} \right) + 2\mathcal{O}_{\phi\theta} \mathbf{p}'^T \mathcal{A}' \mathbf{p}',$$

where

$$\mathcal{A}' = \begin{pmatrix} c_0 + \frac{4c_2m}{(1+m)^2} & c_0 + \frac{2c_2}{1+m} & c_0 \\ c_0 + \frac{2c_2}{1+m} & c_0 & c_0 + \frac{2c_2}{1-m} \\ c_0 & c_0 + \frac{2c_2}{1-m} & c_0 - \frac{4c_2m}{(1-m)^2} \end{pmatrix}.$$

It is then simple to verify that $\mathcal{A}'$ is semipositive definite, which guarantees that $\mathcal{A}$ is positive definite. Thus, the initial ground state, the mean-field symmetry breaking state with a coherent condensate amplitude $\sqrt{N}\phi_j(\vec{r})$, is stable. The associated quantum fluctuations of the atom numbers can be studied by the linearization approximation Eq. (3). We note that $\mathcal{B}$ is diagonalized by an orthogonal matrix

$$\mathcal{U} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}, \quad (23)$$

which motivates the introduction of new number and phase fluctuation operators according to $\mathbf{p}^T = \mathbf{p}'^T \mathcal{U}^T = (p_N, p_M, p_Y)$ and $\mathbf{q}^T = \mathbf{q}'^T \mathcal{U}^T = (q_N, q_M, q_Y)$. Not surprisingly, this transformation recovers the underline symmetry of a spinor-1 condensate, as discovered recently in Ref. [28]. It turns out that p_N , p_M , and p_Y represent, respectively, the fluctuations of the total number of atoms, the magnetization, and the hypercharge. $\mathcal{A}$ is not simultaneously diagonalized by $\mathcal{U}$, yet the zero mode Hamiltonian takes a much simpler form

$$\frac{H_{\text{zero}}}{N} = ap_N^2 + bp_M^2 + cp_Y^2 + \alpha p_N p_Y + \beta p_M p_Y + \gamma p_N p_M + \eta q_Y^2, \quad (24)$$

where all the coefficients are given in Appendix C. By replacing p_j with $-i\hbar \partial/\partial q_j$, we obtain an eigenvalue equation for the distribution of the various fluctuations in the condensate ground state,

$$\begin{aligned} \frac{E}{N\hbar^2} \varphi = & -a \frac{\partial^2 \varphi}{\partial q_N^2} - b \frac{\partial^2 \varphi}{\partial q_M^2} - c \frac{\partial^2 \varphi}{\partial q_Y^2} - \alpha \frac{\partial^2 \varphi}{\partial q_N \partial q_Y} - \beta \frac{\partial^2 \varphi}{\partial q_M \partial q_Y} \\ & - \gamma \frac{\partial^2 \varphi}{\partial q_N \partial q_M} + \frac{\eta}{\hbar^2} q_Y^2 \varphi. \end{aligned} \quad (25)$$

We look for eigenfunctions of the form

$$\chi(q_Y) e^{i(k_N q_N + k_M q_M)}, \quad (26)$$

which, upon substituting into Eq. (25), yields

FIG. 3. The N dependence of Ω/N for the same parameters as in Fig. 1.

$$c \frac{d^2 \chi}{dq_Y^2} + i\kappa \frac{d\chi}{dq_Y} + \left(E' - \frac{\eta}{\hbar^2} q_Y^2 \right) \chi = 0, \quad (27)$$

with

$$\kappa = \alpha k_N + \beta k_M,$$

$$E' = \frac{E}{N\hbar^2} - (ak_N^2 + bk_M^2 + \gamma k_N k_M). \quad (28)$$

Eigenequation (27) is essentially of the harmonic type, which can be easily solved and the resulting eigenvalues of Eq. (25) are

$$\frac{E_n}{N} = (2n+1)\hbar(c\eta)^{1/2} + \Lambda(k_N, k_M),$$

where $n=0, 1, 2, \dots$ and

$$\Lambda(k_N, k_M) = \hbar^2 (ak_N^2 + bk_M^2 + \gamma k_N k_M) - \frac{\hbar^2}{4c} (\alpha k_N + \beta k_M)^2.$$

We note that $\Lambda(k_N, k_M)$ must be greater than or equal to zero due to the positive definiteness of matrix $\mathcal{A}$. Therefore, for the ground state of Eq. (25) we must have $k_N = k_M = 0$, the ground-state ($n=0$) energy and eigenfunction are, respectively,

$$E_0 = N\hbar(c\eta)^{1/2} \equiv \frac{1}{2}\hbar\Omega, \quad (29)$$

$$\varphi_0(q_Y) = \pi^{-1/4} (\hbar\Delta_{Y0})^{-1/2} \exp\left(-\frac{q_Y^2}{2\hbar^2\Delta_{Y0}^2}\right), \quad (30)$$

with

$$\Omega = \frac{2N}{\hbar} [-c_2 \mathcal{O}_{\phi\phi} (\varepsilon - 4c_2 \mathcal{O}_{\phi\theta})]^{1/2}$$

and

$$\Delta_{Y0}^2 = \frac{1}{\hbar} \sqrt{\frac{c}{\eta}} = \frac{1}{3n_0^2} \left(\frac{\varepsilon - 4c_2 \mathcal{O}_{\phi\theta}}{-c_2 \mathcal{O}_{\phi\phi}} \right)^{1/2}.$$

We plotted the selected results for Ω/N and Δ_{Y0}^2 in Figs. 3 and 4, respectively.

FIG. 4. Width Δ_{Y0}^2 of the ground-state distribution for q_Y , as a function of n_0 and N for the same parameters as in Fig. 1 and $\lambda = 1$.

This ground state of H_{zero} can be easily understood. For a system starting with a fixed total number of atoms and magnetization, the dynamic conservation of the total number of atoms and total magnetization, as dictated by Hamiltonian (1), requires $\langle p_N^2 \rangle = \langle p_M^2 \rangle = 0$, which leads to $\langle q_N^2 \rangle = \langle q_M^2 \rangle = \infty$, i.e., the ground state of H_{zero} has completely diffused phases for q_N and q_M ; the inherent ferromagnetic interaction, nevertheless, prepares a correlated ground state such that both p_Y and q_Y take a Gaussian form distribution with $\langle p_Y^2 \rangle = 1/(2\Delta_{Y0}^2)$ and $\langle q_Y^2 \rangle = \hbar^2 \Delta_{Y0}^2 / 2$, which is, in fact, the minimal uncertainty coherent state that is consistent with the symmetry breaking mean-field assumptions of the ground state with three separate macroscopic wave functions for each of the spin components. However, it is experimentally

difficult to produce a condensate having total fixed number and phase fluctuations as specified by such a state. We therefore turn to study the number and phase fluctuations of a general state. We start with the dynamic equations for p_j and q_j ,

$$\dot{p}_N = 0,$$

$$\dot{p}_M = 0,$$

$$\dot{p}_Y = -2N\eta q_Y,$$

$$\dot{q}_N = N(2ap_N + \gamma p_M + \alpha p_Y),$$

$$\dot{q}_M = N(\gamma p_N + 2bp_M + \beta p_Y),$$

$$\dot{q}_Y = N(\alpha p_N + \beta p_M + 2cp_Y). \quad (31)$$

By noting that both $p_N \equiv p_N(0)$ and $p_M \equiv p_M(0)$ are constants of motion, we can express the solutions as

$$\mathbf{x}(t) = \mathcal{T}(t)\mathbf{x}(0), \quad (32)$$

where $\mathbf{x}^T(t) = (p_N(t), p_M(t), p_Y(t), q_N(t), q_M(t), q_Y(t))$, $\mathbf{x}^T(0) = (p_N(0), p_M(0), p_Y(0), q_N(0), q_M(0), q_Y(0))$ is its initial value, and evolution matrix $\mathcal{T}(t)$ takes the following form:

$$\mathcal{T}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\alpha'(1 - \cos \Omega t) & -\beta'(1 - \cos \Omega t) & \cos \Omega t & 0 & 0 & -\sin \Omega t / \hbar \Delta_{Y0}^2 \\ \zeta_a N t + \hbar \Delta_{Y0}^2 \alpha'^2 \sin \Omega t & \zeta_c N t + \hbar \Delta_{Y0}^2 \alpha' \beta' \sin \Omega t & \hbar \Delta_{Y0}^2 \alpha' \sin \Omega t & 1 & 0 & -\alpha'(1 - \cos \Omega t) \\ \zeta_c N t + \hbar \Delta_{Y0}^2 \alpha' \beta' \sin \Omega t & \zeta_b N t + \hbar \Delta_{Y0}^2 \beta'^2 \sin \Omega t & \hbar \Delta_{Y0}^2 \beta' \sin \Omega t & 0 & 1 & -\beta'(1 - \cos \Omega t) \\ \hbar \Delta_{Y0}^2 \alpha' \sin \Omega t & \hbar \Delta_{Y0}^2 \beta' \sin \Omega t & \hbar \Delta_{Y0}^2 \sin \Omega t & 0 & 0 & \cos \Omega t \end{pmatrix}. \quad (33)$$

Coefficients α' , β' , ζ_a , ζ_b , and ζ_c are given in Appendix D. The explicit solutions of the time-dependent number and phase fluctuations are given in Appendix E. We note that the general solution has a simple structure: in addition to oscillating terms of form $\cos \Omega t$ and $\sin \Omega t$, the phase fluctuations of q_N and q_M also contain terms proportional to Nt . This indicates that our linearization approximation Eq. (3) is valid only for a finite duration. The time-dependent covariance can also be obtained to be

$$\langle x_i(t) x_j(t) \rangle = \sum_{kl} \mathcal{T}_{ik}(t) \mathcal{T}_{jl}(t) \langle x_k(0) x_l(0) \rangle. \quad (34)$$

Finally, let us consider the diffusion of the direction of macroscopic spin $\vec{f}(\vec{r}) \equiv \sum_{ij} \psi_j^\dagger(\vec{r}) \vec{\mathbf{F}}_{ij} \psi_j(\vec{r})$. As we have shown in Ref. [20], independent of the spatial coordinates, the spins of Bose condensed atoms are parallel for ferromag-

netic interactions, i.e., it essentially acts as a macroscopic magnetic dipole pointing along the same direction,

$$\vec{f}_0(\vec{r}) = N \sum_{ij} \phi_i^*(\vec{r}) \vec{\mathbf{F}}_{ij} \phi_j(\vec{r}) = N \phi^2(\vec{r}) \begin{pmatrix} \sqrt{1-m^2} \cos \Theta \\ -\sqrt{1-m^2} \sin \Theta \\ m \end{pmatrix}, \quad (35)$$

where $\Theta = \alpha_+ - \alpha_0 = \alpha_0 - \alpha_-$. Our initial choice of three separate macroscopic condensate wave functions (with given phases) for each of the spin components fixes the initial direction of the condensate spin direction. As the phase spreading dynamics attempts to restore the U(1) symmetries of each of the phases, our initial choice of phases will become irrelevant. Under linear approximation (3), the fluctuation becomes

$$\delta \vec{f}(t) = \int d\vec{r} [\vec{f}(\vec{r}) - \vec{f}_0(\vec{r})], \quad (36)$$

or, explicitly,

$$\begin{aligned} \delta f_x &= \frac{1}{\sqrt{2}} \int d\vec{r} (\phi_0^* \delta \psi_+ + \phi_+ \delta \psi_0^\dagger + \phi_-^* \delta \psi_0 + \phi_0 \delta \psi_-^\dagger \\ &\quad + \text{H.c.}), \\ \delta f_y &= \frac{i}{\sqrt{2}} \int d\vec{r} (\phi_0^* \delta \psi_+ + \phi_+ \delta \psi_0^\dagger + \phi_-^* \delta \psi_0 + \phi_0 \delta \psi_-^\dagger \\ &\quad - \text{H.c.}), \\ \delta f_z &= \int d\vec{r} (\phi_+^* \delta \psi_+ - \phi_-^* \delta \psi_- + \text{H.c.}). \end{aligned} \quad (37)$$

Although $\langle \delta \vec{f}(t) \rangle = 0$, the dynamics of the zero mode causes the variance of $\delta \vec{f}(t)$ to be nonzero. It can be re-expressed in terms of the (time-dependent) number and phase fluctuation operators, under the RWA, as

$$\begin{aligned} \delta f_x(t) &= \frac{\cos \Theta}{\sqrt{1-m^2}} [\sqrt{3} p_N(t) - \sqrt{2} m p_M(t)] \\ &\quad + \frac{\sqrt{1-m^2} \sin \Theta}{2\hbar} [\sqrt{2} q_M(t) + \sqrt{6} m q_Y(t)], \\ \delta f_y(t) &= \frac{-\sin \Theta}{\sqrt{1-m^2}} [\sqrt{3} p_N(t) - \sqrt{2} m p_M(t)] \\ &\quad + \frac{\sqrt{1-m^2} \cos \Theta}{2\hbar} [\sqrt{2} q_M(t) + \sqrt{6} m q_Y(t)], \\ \delta f_z(t) &= \sqrt{2} p_M(t), \end{aligned} \quad (38)$$

where the time-dependent terms are

$$\begin{aligned} \sqrt{2} q_M(t) + \sqrt{6} m q_Y(t) &= \sqrt{2} q_M(0) + \sqrt{6} m q_Y(0) + \frac{4\epsilon N t}{1-m^2} \\ &\quad \times [-\sqrt{3} m p_N(0) + \sqrt{2} p_M(0)]. \end{aligned} \quad (39)$$

We see that δf_z is a constant of motion due to the conservation of total magnetization. On the other hand, both δf_x and δf_y diffuse with time. To calculate the variance of $\delta \vec{f}(t)$, we simply take $\Theta = 0$, which yields

$$\begin{aligned} \delta f_x(t) &= \frac{1}{\sqrt{1-m^2}} [\sqrt{3} p_N(0) - \sqrt{2} m p_M(0)], \\ \delta f_y(t) &= \frac{\sqrt{1-m^2}}{2\hbar} [\sqrt{2} q_M(t) + \sqrt{6} m q_Y(t)], \end{aligned}$$

FIG. 5. Illustration of the macroscopic condensate spin and its fluctuation in spherical coordinates.

$$\delta f_z(t) = \sqrt{2} p_M(0), \quad (40)$$

i.e., with the direction of the initial macroscopic spin constrained in the xz plane. The results of the variances for condensate spin direction fluctuations are listed in Appendix F.

We can also project the fluctuation of the spin [Eq. (38)] onto spherical coordinates $(\hat{r}, \hat{\theta}, \hat{\phi})$, as illustrated in Fig. 5, and noting that

$$\begin{aligned} \hat{r} &= \hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta, \\ \hat{\theta} &= \hat{x} \cos \theta \cos \phi + \hat{y} \cos \theta \sin \phi - \hat{z} \sin \theta, \\ \hat{\phi} &= -\hat{x} \sin \phi + \hat{y} \cos \phi, \end{aligned} \quad (41)$$

and

$$\begin{aligned} \sin \theta &= \sqrt{1-m^2}, \quad \cos \theta = m, \\ \sin \phi &= -\sin \Theta, \quad \cos \phi = \cos \Theta, \end{aligned}$$

we find

$$\begin{aligned} \delta f_r(t) &= \sqrt{3} p_N(0), \\ \delta f_\theta(t) &= \frac{1}{\sqrt{1-m^2}} [\sqrt{3} m p_N(0) - \sqrt{2} p_M(0)], \\ \delta f_\phi(t) &= \frac{\sqrt{1-m^2}}{2\hbar} [\sqrt{2} q_M(t) + \sqrt{6} m q_Y(t)]. \end{aligned} \quad (42)$$

We immediately see that both $\delta f_r(t) \equiv \delta f_r(0)$ and $\delta f_\theta(t) \equiv \delta f_\theta(0)$ are fixed due to the conservations of N and $\mathcal{M}$. Furthermore, with the use of Eq. (39), we find that $\delta f_\phi(t)$ can be rewritten as

$$\delta f_\phi(t) = \delta f_\phi(0) - \frac{2\epsilon N t}{\hbar} \delta f_\theta(0), \quad (43)$$

where

$$\delta f_\phi(0) = \frac{\sqrt{1-m^2}}{2\hbar} [\sqrt{2}q_M(0) + \sqrt{6}mq_Y(0)].$$

We then see that $\delta f_\phi(t)$ grows linearly with t and the diffusion rate is given by

$$R_d = -\frac{2\varepsilon N}{\hbar} \delta f_\theta(0), \quad (44)$$

which is proportional to $\delta f_\theta(0)$. A result easily understandable in terms of the single axis twisting of isospin T_z^2 (of the Hamiltonian) of a spinor-1 condensate [28].

Given any initial condensate state, its variances in atom numbers, for each individual spin component and their correlations, and related phase fluctuations completely determine the subsequent fluctuations of $\delta f_\phi(t)$. $P_j(0)$ is, in fact, nothing but the atom number fluctuation of state $|j\rangle$. In general, we note that

$$\begin{aligned} \int d\vec{r} \psi_j^\dagger(\vec{r}) \psi_j(\vec{r}) &= N + \sqrt{N} P_j + \int d\vec{r} \delta \psi_j^\dagger(\vec{r}) \delta \psi_j(\vec{r}) \\ &= N + \sqrt{N} P_j + \frac{sn_j}{2J_j^2} P_j^2 + \frac{n_j}{2\hbar^2 J_j^2} Q_j^2 - \frac{1}{2}, \end{aligned} \quad (45)$$

where $s = \int d\vec{r} \theta^2(\vec{r})$ and we have used RWA in deriving the last line. One can easily find

$$\begin{aligned} \delta N_j^2 &= \langle N_j^2 \rangle - \langle N_j \rangle^2 \\ &= \left\langle \left[\int d\vec{r} \psi_j^\dagger(\vec{r}) \psi_j(\vec{r}) \right]^2 \right\rangle - \left\langle \int d\vec{r} \psi_j^\dagger(\vec{r}) \psi_j(\vec{r}) \right\rangle^2 \\ &\approx N \langle P_j^2 \rangle \\ &= N \langle P_j'^2 \rangle, \end{aligned} \quad (46)$$

to lowest order.

If we use σ_N^2 , σ_M^2 , and σ_Y^2 to denote the variances of the total atom number (N), the magnetization ($\mathcal{M}$), and the hypercharge (Y), we find that

$$\langle [\delta f_r(t)]^2 \rangle = 3\sigma_N^2,$$

$$\langle [\delta f_\theta(t)]^2 \rangle = \frac{1}{1-m^2} (3m^2\sigma_N^2 + 2\sigma_M^2). \quad (47)$$

To calculate the variance of $\delta f_\phi(t)$, we assume a completely uncorrelated (between fluctuations of N , $\mathcal{M}$, and Y) initial distribution in the form of a Gaussian wave packet

$$\begin{aligned} \varphi(p_N, p_M, p_Y) &= (2\pi)^{-3/4} (\sigma_N \sigma_M \sigma_Y)^{-1/2} \\ &\times \exp \left[-\frac{1}{4} \left(\frac{p_N^2}{\sigma_N^2} + \frac{p_M^2}{\sigma_M^2} + \frac{p_Y^2}{\sigma_Y^2} \right) \right]. \end{aligned} \quad (48)$$

Since $q_j = i\hbar \partial / \partial p_j$, we find, for such a distribution,

$$\langle q_i(0) q_j(0) \rangle = \frac{\hbar^2}{4\sigma_i^2} \delta_{ij},$$

$$\langle p_i(0) p_j(0) \rangle = \sigma_i^2 \delta_{ij},$$

$$\langle q_i(0) p_i(0) + p_i(0) q_i(0) \rangle = 0,$$

$$\langle q_i(0) p_j(0) \rangle = 0 \quad \text{for } i \neq j, \quad (49)$$

and $i, j = N, M, Y$. Therefore,

$$\langle [\delta f_\phi(t)]^2 \rangle = \langle [\delta f_\phi(0)]^2 \rangle + \frac{4\varepsilon^2 N^2 t^2}{\hbar^2} \langle [\delta f_\theta(0)]^2 \rangle, \quad (50)$$

with

$$\langle [\delta f_\phi(0)]^2 \rangle = \frac{1-m^2}{8} \left(\frac{1}{\sigma_M^2} + 3m^2 \frac{1}{\sigma_Y^2} \right). \quad (51)$$

If, on the other hand, we take a distribution similar to Eq. (48) but with no correlations between the populations in the spin components $j = +, 0, -$,

$$\begin{aligned} \varphi(p'_+, p'_0, p'_-) &= (2\pi)^{-3/4} (\sigma_+ \sigma_0 \sigma_-)^{-1/2} \\ &\times \exp \left[-\frac{1}{4} \left(\frac{p'^2_+}{\sigma_+^2} + \frac{p'^2_0}{\sigma_0^2} + \frac{p'^2_-}{\sigma_-^2} \right) \right], \end{aligned} \quad (52)$$

with σ_j^2 denoting the corresponding atom number variance, we find

$$\langle [\delta f_r(t)]^2 \rangle = \sigma_+^2 + \sigma_0^2 + \sigma_-^2,$$

$$\langle [\delta f_\theta(t)]^2 \rangle = \frac{1}{1-m^2} [(1-m)^2 \sigma_+^2 + m^2 \sigma_0^2 + (1+m)^2 \sigma_-^2], \quad (53)$$

and Eq. (50) still holds but with

$$\langle [\delta f_\phi(0)]^2 \rangle = \frac{1-m^2}{16} \left[\frac{(1+m)^2}{\sigma_+^2} + 4m^2 \frac{1}{\sigma_0^2} + \frac{(1-m)^2}{\sigma_-^2} \right]. \quad (54)$$

In the Thomas-Fermi (TF) regime, we obtain most of the above results analytically. First we note that the ground wave function is

$$\phi^2(\vec{r}) = \begin{cases} (c_+ N)^{-1} [\mu - V_{\text{ext}}(\vec{r})] & \text{if } \mu > V_{\text{ext}}(\vec{r}), \\ 0 & \text{otherwise,} \end{cases} \quad (55)$$

where $c_+ = 4\pi\hbar^2 a_2 / M$ and chemical potential

$$\mu = \frac{\hbar \omega_{\text{ho}}}{2} \left(\frac{15 N a_2}{a_{\text{ho}}} \right)^{2/5}$$

with $\omega_{\text{ho}} = (\omega_x \omega_y \omega_z)^{1/3}$ and $a_{\text{ho}} = (\hbar / M \omega_{\text{ho}})^{1/2}$. Similarly, ignoring the kinetic-energy term in Eq. (18), we obtain

$$\theta(\vec{r}) = \begin{cases} [2V_0 \phi(\vec{r})]^{-1} & \text{if } \mu > V_{\text{ext}}(\vec{r}), \\ 0 & \text{otherwise} \end{cases} \quad (56)$$

and

$$\tilde{u} = \frac{c_+}{V_0} \quad (57)$$

with

$$V_0 = \int_{\mu - V_{\text{ext}}(\vec{r}) > 0} d\vec{r} = \frac{4\pi a_{\text{ho}}^3}{3} \left(\frac{15a_2}{a_{\text{ho}}} \right)^{3/5} N^{3/5}.$$

All overlap integrals can be calculated analytically as

$$\begin{aligned} \mathcal{O}_{\phi\theta} &= (4V_0)^{-1} = \frac{3}{16\pi a_{\text{ho}}^3} \left(\frac{a_{\text{ho}}}{15a_2} \right)^{3/5} N^{-3/5}, \\ \mathcal{O}_{\phi\phi} &= \frac{32\pi\mu^{7/2}}{105c_+^2 N^2 \omega_{\text{ho}}^3} \left(\frac{2}{M} \right)^{3/2} = \frac{15^{2/5}}{14\pi a_{\text{ho}}^3} \left(\frac{a_{\text{ho}}}{a_2} \right)^{3/5} N^{-3/5}. \end{aligned}$$

Parameter ε relates essentially to kinetic-energy operator $E_k^{(\theta)} = -(\hbar^2/2M) \int d\vec{r} \theta \nabla^2 \theta$, evaluated with respect to phase function $\theta(\vec{r})$. In the TF approximation we find

$$N\varepsilon = 2E_k^{(\theta)}. \quad (58)$$

Using Eqs. (15) and (18) we find

$$(\mathcal{L} - \mu + 3c_+ N \phi^2) \theta = \frac{N\tilde{u}}{\mu} (\mathcal{L} + c_+ N \phi^2) \phi, \quad (59)$$

which simplifies to

$$\left(-\frac{\hbar^2 \nabla^2}{2M} + 2c_+ N \phi^2 \right) \theta = \frac{N\tilde{u}}{\mu} \left(-\frac{\hbar^2 \nabla^2}{2M} + \mu \right) \phi, \quad (60)$$

in the TF limit. Multiplying Eq. (60) from left by θ and ϕ , respectively, and integrating over $\vec{r}$, we obtain

$$\begin{aligned} E_k^{(\theta)} + 2c_+ N \mathcal{O}_{\phi\theta} &= \frac{N\tilde{u}}{\mu} \left(E_k^{(\phi\theta)} + \frac{\mu}{2} \right), \\ E_k^{(\phi\theta)} + 2c_+ N \int d\vec{r} \phi^3 \theta &= \frac{N\tilde{u}}{\mu} (E_k^{(\phi)} + \mu), \end{aligned} \quad (61)$$

where $E_k^{(\phi)} = -(\hbar^2/2M) \int d\vec{r} \phi \nabla^2 \phi$ is the kinetic energy per atom in the condensate and $E_k^{(\phi\theta)} = -(\hbar^2/2M) \int d\vec{r} \phi \nabla^2 \theta$ $= -(\hbar^2/2M) \int d\vec{r} \theta \nabla^2 \phi$. Eliminating $E_k^{(\phi\theta)}$ from above two equations, we obtain

$$\begin{aligned} E_k^{(\theta)} &= \left(\frac{N\tilde{u}}{\mu} \right)^2 E_k^{(\phi)} + \frac{N\tilde{u}}{2} - 2c_+ N \mathcal{O}_{\phi\theta} \\ &\quad + \frac{N\tilde{u}}{\mu} \left(N\tilde{u} - 2c_+ N \int d\vec{r} \phi^3 \theta \right) \\ &= \left(\frac{N\tilde{u}}{\mu} \right)^2 E_k^{(\phi)} \\ &= \frac{4}{25} E_k^{(\phi)}. \end{aligned} \quad (62)$$

For a spherical trap, adopting the result of Ref. [29], we immediately find

$$\varepsilon = \frac{4\hbar^2}{5MNR^2} \ln \left(\frac{R}{1.3a_{\text{ho}}} \right), \quad (63)$$

with

$$R = (15a_2 a_{\text{ho}}^4)^{1/5} N^{1/5},$$

which seems to indicate that $N\varepsilon \propto N^{-2/5}$ asymptotically goes to zero at large N . By all measures, we find ε to be a very small quantity as compared to $c_2 \mathcal{O}_{\phi\theta}$. Taking ε as zero in the TF limit, we obtain

$$\begin{aligned} n_0^2 \Delta_{Y0}^2 &= \frac{1}{3} \sqrt{\frac{7}{10}}, \\ \hbar \Omega &= -\frac{\sqrt{3/14}}{15^{1/10}} \left(\frac{c_2}{\pi a_{\text{ho}}^3} \right) \left(\frac{a_{\text{ho}}}{a_2} \right)^{3/5} N^{2/5}. \end{aligned} \quad (64)$$

To shed light on the above scaling results based on the TF approximation, we have performed extensive numerical calculations. As an example, we consider a spinor condensate of ^{87}Rb atoms in a spherically symmetric trap with a harmonic frequency $\omega = (2\pi)100$ (Hz) and consider the regime of $N \sim 10^7 - 10^{15}$. It turns out that our results indeed confirm the scaling relationship

$$\mathcal{O}_{\phi\phi}, \mathcal{O}_{\phi\theta}, \tilde{u}, \Omega / N \propto N^{-3/5},$$

and

$$\begin{aligned} \frac{N\tilde{u}}{\mu} &\rightarrow \frac{2}{5}, \\ n_0^2 \Delta_{Y0}^2 &\rightarrow \frac{1}{3} \sqrt{\frac{7}{10}}, \end{aligned} \quad (65)$$

in the large- N limit. We also find that

$$E_k^{(\phi)} \propto N^{-0.3549},$$

in rough agreement with the result of Ref. [29]; On the other hand, we find

$$N\varepsilon \propto N^{0.1348}, \quad E_k^{(\theta)} \propto N^{0.1314},$$

and

$$N\varepsilon \approx 4E_k^{(\theta)},$$

which do not agree with the results above, as obtained with the TF approximation. A careful analysis, as confirmed by our numerical results, reveals that although $\int d\vec{r}(V_{\text{ext}} - \mu + c_+ N \phi^2) \phi^2$ is small, $\int d\vec{r}(V_{\text{ext}} - \mu + c_+ N \phi^2) \theta^2$ is not, and, approximately, we find

$$E_k^{(\theta)} \approx \frac{1}{2} \int d\vec{r} (V_{\text{ext}} - \mu + c_+ N \phi^2) \theta^2. \quad (66)$$

This leads to the opposite N dependence for $E_k^{(\theta)}$ and $E_k^{(\phi)}$; while $E_k^{(\theta)}$ increases with increasing values of N , $E_k^{(\phi)}$ decreases. Thus, identity Eq. (62), as obtained in the TF limit, is invalid.

IV. A NONZERO MAGNETIC FIELD

Inside a nonzero homogeneous magnetic field ($\vec{B}$), the atomic quantization axis becomes fixed, along the direction of $\vec{B}$, or more conveniently denoted as the z axis. Mathematically, it turns out that this is equivalent to the requirement of the conservation of magnetization $\mathcal{M} = N_+ - N_-$. This reduces the SO(3) rotational symmetry to its subgroup SO(2), which is still a continuous symmetry. Thus, the quantum dynamics of the phase remains formally the same as discussed before.

In the linear Zeeman regime, the original Hamiltonian (1) is augmented to

$$H' = H - \hbar \omega_L \mathbf{F}_z. \quad (67)$$

The Larmor precession frequency is $\omega_L = B \mu_B / \hbar$ with μ_B being the magnetic dipole moment for state $|F=1, M_F=1\rangle$. In this case, the only change is the phases of the spatial wave functions and the phase functions, if of the following time-dependent form:

$$\alpha_j = \alpha_{j0} + j \omega_L t \quad (68)$$

for $j = +, 0$, and $-$, $\alpha_{+0} - \alpha_{00} = \alpha_{00} - \alpha_{-0} = 0$ still holds true for a ferromagnetic spinor-1 condensate. A direct consequence of the magnetic field is thus a time-dependent relative phase

$$\Theta(t) = \alpha_{+0} - \alpha_{00} + \omega_L t = \alpha_{00} - \alpha_{-0} + \omega_L t. \quad (69)$$

When applied to the macroscopic condensate spin, we find that the spin gains a precession around the z axis.

Now the picture of the dynamics of the macroscopic condensate spin is clear: besides the precession induced by the external B field, it also diffuses to restore the U(1) and SO(2) symmetries of the Hamiltonian.

The effect of quadratic Zeeman shift can also be simply addressed. Apart from an overall shift, it introduces a (positive) B -field dependent level shift $\propto \hbar \delta_B F_z^2$ to states $|\pm\rangle$ with

respect to state $|0\rangle$. This differential shift causes the precessing of the condensate spin to be twisted, i.e., the precessing of the $|+\rangle$ ($|-\rangle$) component being slower (faster) by δ_B .

V. THE CASE OF ANTIFERROMAGNETIC INTERACTIONS

Our discussion of the phase diffusion in the preceding section is based on the proof that the single spatial mode approximation is exact for ferromagnetic interactions. One might, therefore, conclude that a similar study for antiferromagnetic interactions would be difficult as, unlike for ferromagnetic interactions, it is difficult to predict the ground-state condensate structure, although it is clear that the SMA becomes invalid, in general [20]. Under the SMA, other studies [30,31] have indicated that the ground state could become fragmented when the external magnetic field is negligibly small. Adopting the mean-field ground state, however, the quantum phase problem for antiferromagnetic interactions is much simpler.

First we note that when $\mathcal{M} \neq 0$, the population in the 0th component is zero in the ground state [20,24]. Therefore, the zero mode Hamiltonian (20) reduces to

$$\begin{aligned} \frac{H_{\text{zero}}}{N} = & -\frac{P_+^2}{J_+^2} \left(-\frac{J_+}{2} \tilde{u} + 2c_- I_{-+} \right) - \frac{P_-^2}{J_-^2} \left(-\frac{J_-}{2} \tilde{u} \right. \\ & \left. + 2c_- I_{-+} \right) + \frac{4c_- P_+ P_-}{J_+ J_-} I_{-+}, \end{aligned} \quad (70)$$

which is exactly the same zero mode Hamiltonian for a binary condensate with no coupling between its two components. This has already been studied [7]. We note that the results obtained in Sec. II remain valid.

Now, let's consider the $\mathcal{M}=0$ case. As was pointed out in Ref. [20], the single mode approximation again applies in this case, and we can assume

$$\phi_j(\vec{r}) = \sqrt{n_j} \phi(\vec{r}) e^{i\alpha_j}, \quad (71)$$

where $n_+ = n_- = (1 - n_0)/2$ and total energy of the system is independent of the value of n_0 . A similar relation exists among the phases of the three components: $\alpha_+ + \alpha_- - 2\alpha_0 = \pi$. Spatial profile $\phi(\vec{r})$ satisfies the following equation:

$$[\mathcal{L} + c_0 N \phi^2(\vec{r})] \phi(\vec{r}) = \mu \phi(\vec{r}), \quad (72)$$

with a coefficient c_0 for the nonlinear term. The phase function then becomes

$$\theta_j(\vec{r}) = \sqrt{n_j} \theta(\vec{r}) e^{i\alpha_j}, \quad (73)$$

where $\theta(\vec{r})$ is the solution of the following equation:

$$[\mathcal{L} - \mu + 3c_0 N \phi^2(\vec{r})] \theta(\vec{r}) = N \tilde{u} \theta(\vec{r}). \quad (74)$$

Following the results of Sec. III, we define

$$\varepsilon = \frac{2}{N} \int d\vec{r} \theta(\vec{r}) [\mathcal{L} - \mu + c_0 N \phi^2(\vec{r})] \theta(\vec{r}), \quad (75)$$

and the zero mode Hamiltonian becomes

$$\begin{aligned}
 \frac{H_{\text{zero}}}{N} = & \left[\frac{\varepsilon}{(1-n_0)\mathcal{O}_{\phi\theta}} + 2c_0 + \frac{2c_2}{1-n_0} \right] p'^2_{+} \mathcal{O}_{\phi\theta} + \left(\frac{\varepsilon}{2n_0\mathcal{O}_{\phi\theta}} \right. \\
 & \left. + 2c_0 \right) p'^2_{-} \mathcal{O}_{\phi\theta} + \left[\frac{\varepsilon}{(1-n_0)\mathcal{O}_{\phi\theta}} + 2c_0 \right. \\
 & \left. + \frac{2c_2}{1-n_0} \right] p'^2_{-} \mathcal{O}_{\phi\theta} + 4c_0 p'_+ p'_0 \mathcal{O}_{\phi\theta} + 4c_0 p'_- p'_0 \mathcal{O}_{\phi\theta} \\
 & + 4 \left(c_0 + c_2 \frac{2n_0-1}{1-n_0} \right) p'_+ p'_- \mathcal{O}_{\phi\theta} \\
 & + \frac{c_2 n_0 (1-n_0) \mathcal{O}_{\phi\phi}}{2\hbar^2} (q'_+ - 2q'_0 + q'_-)^2 \\
 = & \mathbf{p}'^T \mathcal{A} \mathbf{p}' + \mathbf{q}'^T \mathcal{B} \mathbf{q}'. \quad (76)
 \end{aligned}$$

The positive definiteness of matrices $\mathcal{A}$ and $\mathcal{B}$ can again be verified. By applying the same transformation (23), we obtain

$$\frac{H_{\text{zero}}}{N} = ap_N^2 + bp_M^2 + cp_Y^2 + \alpha p_N p_Y + \eta q_Y^2, \quad (77)$$

where all coefficients are now given in Appendix G. We see that fluctuations in N are completely decoupled from that of

M and Y in this case, a result that also happens for ferromagnetic interactions when $\mathcal{M}=0$. The ground state of Eq. (76) is similar to Eq. (20), except that now

$$\begin{aligned}
 \Omega = 2N\sqrt{c\eta} = \frac{2N}{\hbar} [c_2 \mathcal{O}_{\phi\phi} (\varepsilon + 4c_2 n_0^2 \mathcal{O}_{\phi\theta})]^{1/2}, \\
 \Delta_{Y0}^2 = \frac{1}{\hbar} \sqrt{\frac{c}{\eta}} = \frac{1}{3} \left[\frac{\varepsilon + 4c_2 n_0^2 \mathcal{O}_{\phi\theta}}{c_2 n_0^2 (1-n_0)^2 \mathcal{O}_{\phi\phi}} \right]^{1/2}. \quad (78)
 \end{aligned}$$

The dynamic equations for p_i and q_i then become

$$\begin{aligned}
 \dot{p}_N &= 0, \\
 \dot{p}_M &= 0, \\
 \dot{p}_Y &= -2N\eta q_Y, \\
 \dot{q}_N &= N(2ap_N + \alpha p_Y), \\
 \dot{q}_M &= 2Nb p_M, \\
 \dot{q}_Y &= N(\alpha p_N + 2cp_Y). \quad (79)
 \end{aligned}$$

One can easily find the evolution matrix to be

$$\mathcal{T}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\alpha'(1-\cos\Omega t) & 0 & \cos\Omega t & 0 & 0 & -\sin\Omega t/\hbar\Delta_{Y0}^2 \\ \zeta_a Nt + \hbar\Delta_{Y0}^2 \alpha'^2 \sin\Omega t & 0 & \hbar\Delta_{Y0}^2 \alpha' \sin\Omega t & 1 & 0 & -\alpha'(1-\cos\Omega t) \\ 0 & 2bNt & 0 & 0 & 1 & 0 \\ \hbar\Delta_{Y0}^2 \alpha' \sin\Omega t & 0 & \hbar\Delta_{Y0}^2 \sin\Omega t & 0 & 0 & \cos\Omega t \end{pmatrix}, \quad (80)$$

where

$$\begin{aligned}
 \alpha' &= \frac{\alpha}{2c} = \frac{(3n_0-1)\varepsilon + 8c_2 n_0^2 \mathcal{O}_{\phi\theta}}{\sqrt{2}(\varepsilon + 4c_2 n_0^2 \mathcal{O}_{\phi\theta})}, \\
 \zeta_a &= \frac{4ac - \alpha^2}{2c} = 12c_0 \mathcal{O}_{\phi\theta} + \frac{3\varepsilon(\varepsilon + 4c_2 n_0^2 \mathcal{O}_{\phi\theta})}{\varepsilon + 4c_2 n_0^2 \mathcal{O}_{\phi\theta}}.
 \end{aligned}$$

As for $\delta\vec{f}(t)$, we find

$$\begin{aligned}
 \delta f_x &= \cos\Theta \sqrt{\frac{2n_0}{1-n_0}} p_M + \sin\Theta \frac{\sqrt{6n_0(1-n_0)}}{\hbar} q_Y, \\
 \delta f_y &= -\sin\Theta \sqrt{\frac{2n_0}{1-n_0}} p_M + \cos\Theta \frac{\sqrt{6n_0(1-n_0)}}{\hbar} q_Y, \\
 \delta f_z &= \sqrt{2} p_M. \quad (81)
 \end{aligned}$$

VI. CONCLUSIONS

In conclusion, we have investigated, in detail, quantum phase diffusions of a spinor-1 condensate. When the elastic atomic interaction is of ferromagnetic type, the structure of the ground state is greatly simplified: the spatial mode functions of the different spin components are exactly the same and accordingly, we also proved that the phase functions are exactly the same. This simplification allows us to construct the zero mode Hamiltonian which describes the number and phase fluctuations of a condensate while maintaining the conservations of the total number of atoms and the total magnetization. We have provided analytical results for the number and phase fluctuations due to both the quantum phase diffusion dynamics and the initial distribution of atom number and phase fluctuations. The structure of these fluctuations is rather simple: along with the oscillations due to $\sin\Omega t$ and $\cos\Omega t$ terms, quantum phase diffusion terms proportional to Nt also exist. This suggests that the mean-field

approach to spinor-1 condensates is only valid for a finite duration. Based on these results, we have studied the diffusion of the direction of a macroscopic condensate spin. Our investigation sheds important light on the studies of quantum correlations, beyond mean-field theory, among Bose condensed atoms.

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APPENDIX B: THE ZERO MODE HAMILTONIAN

The zero mode Hamiltonian is listed below.

$$\begin{aligned} \frac{H_{\text{zero}}}{N} = & -\frac{P_+^2}{J_+^2} \left[-\frac{J_+}{2} \tilde{u} + 2c_- I_{-+} + 2c_+ I_{+0} + 2c_2 (\phi_0 \theta_0 | \phi_- \theta_+)' + c_2 (\phi_0^2 | \theta_+ \theta_-)' \right] - \frac{2P_0^2}{J_0^2} \left[-\frac{J_0}{4} \tilde{u} + c_+ I_{+0} + c_+ I_{-0} \right. \\ & \left. + c_2 (\phi_- \theta_+ | \phi_0 \theta_0)' + c_2 (\phi_+ \theta_- | \phi_0 \theta_0)' \right] - \frac{P_-^2}{J_-^2} \left[-\frac{J_-}{2} \tilde{u} + 2c_- I_{-+} + 2c_+ I_{-0} + 2c_2 (\phi_0 \theta_0 | \phi_+ \theta_-)' + c_2 (\phi_0^2 | \theta_+ \theta_-)' \right] \\ & + \frac{4P_+ P_0}{J_+ J_0} [c_+ I_{+0} + c_2 (\phi_0 \theta_0 | \phi_- \theta_+)] + \frac{4P_- P_0}{J_- J_0} [c_+ I_{-0} + c_2 (\phi_0 \theta_0 | \phi_+ \theta_-)] + \frac{2P_+ P_-}{J_+ J_-} [2c_- I_{-+} + c_2 (\phi_0^2 | \theta_+ \theta_-)] \\ & - \frac{c_2}{\hbar^2} (\phi_0^2 | \phi_+ \phi_-)' \left(\frac{Q_+}{J_+} - \frac{2Q_0}{J_0} + \frac{Q_-}{J_-} \right)^2. \end{aligned} \quad (\text{B1})$$

APPENDIX C: COEFFICIENTS IN THE ZERO MODE HAMILTONIAN FOR FERROMAGNETIC INTERACTIONS

The coefficients in the zero mode Hamiltonian (24) are

$$\begin{aligned} a = & \frac{1}{4n_0^2} \left[\frac{\varepsilon}{3} (5 + 3m^2) + 2\mathcal{O}_{\phi\theta} \left(3c_0(1 - m^2)^2 + \frac{8}{3}c_2(1 - 3m^2) \right) \right], \quad b = \frac{1}{2n_0^2} [\varepsilon(1 + m^2) - 8c_2\mathcal{O}_{\phi\theta}m^2], \\ c = & \frac{1}{3n_0^2} (\varepsilon - 4c_2\mathcal{O}_{\phi\theta}), \quad \alpha = \frac{\sqrt{2}}{6n_0^2} (1 + 3m^2)(\varepsilon - 4c_2\mathcal{O}_{\phi\theta}), \quad \beta = -\frac{2m}{\sqrt{3}n_0^2} (\varepsilon - 4c_2\mathcal{O}_{\phi\theta}), \\ \gamma = & -\frac{2m}{n_0^2} \sqrt{\frac{2}{3}} [\varepsilon - c_2\mathcal{O}_{\phi\theta}(1 + 3m^2)], \quad \eta = -3c_2n_0^2\mathcal{O}_{\phi\phi}/\hbar^2. \end{aligned} \quad (\text{C1})$$

APPENDIX D: PARAMETERS IN EVOLUTION MATRIX

The list of parameters in evolution matrix $\mathcal{T}(t)$, of Eq. (33) are

$$\begin{aligned} \alpha' = & \frac{\alpha}{2c} = \frac{1 + 3m^2}{2\sqrt{2}}, \quad \beta' = \frac{\beta}{2c} = -\sqrt{3}m, \quad \zeta_a = \frac{4ac - \alpha^2}{2c} = \frac{3(1 + m^2)\varepsilon}{1 - m^2} + 12(c_0 + c_2)\mathcal{O}_{\phi\theta}, \\ \zeta_b = & \frac{4bc - \beta^2}{2c} = \frac{4\varepsilon}{1 - m^2}, \quad \zeta_c = \frac{2c\gamma - \alpha\beta}{2c} = -\frac{2\sqrt{6}m\varepsilon}{1 - m^2}. \end{aligned} \quad (\text{D1})$$

APPENDIX E: EXPLICIT FORM OF NUMBER AND PHASE FLUCTUATION

Here we present the explicit form of the solutions for the number and phase fluctuations.

$$p_N(t) = p_N(0), \quad p_M(t) = p_M(0),$$

$$p_Y(t) = -\frac{1+3m^2}{2\sqrt{2}}(1-\cos\Omega t)p_N(0) + \sqrt{3}m(1-\cos\Omega t)p_M(0) + p_Y(0)\cos\Omega t - \frac{q_Y(0)}{\hbar\Delta_{Y0}^2}\sin\Omega t,$$

$$q_N(t) = \left[Nt \left(\frac{3(1+m^2)\varepsilon}{1-m^2} + 12(c_0+c_2)\mathcal{O}_{\phi\theta} \right) + \frac{(1+3m^2)^2}{8}\hbar\Delta_{Y0}^2\sin\Omega t \right] p_N(0) - \left[Nt \frac{2\sqrt{6}m\varepsilon}{1-m^2} \right. \\ \left. + \frac{\sqrt{3}m(1+3m^2)}{2\sqrt{2}}\hbar\Delta_{Y0}^2\sin\Omega t \right] p_M(0) + \frac{1+3m^2}{2\sqrt{2}}\hbar^2\Delta_{Y0}^2p_Y(0)\sin\Omega t + q_N(0) - \frac{1+3m^2}{2\sqrt{2}}(1-\cos\Omega t)q_Y(0),$$

$$q_M(t) = - \left[Nt \frac{2\sqrt{6}m\varepsilon}{1-m^2} + \hbar\Delta_{Y0}^2 \frac{\sqrt{3}m(1+3m^2)}{2\sqrt{2}}\sin\Omega t \right] p_N(0) + \left(Nt \frac{4\varepsilon}{1-m^2} + 3m^2\hbar\Delta_{Y0}^2\sin\Omega t \right) p_M(0) \\ - \sqrt{3}m\hbar\Delta_{Y0}^2p_Y(0)\sin\Omega t + q_M(0) + \sqrt{3}m(1-\cos\Omega t)q_Y(0),$$

$$q_Y(t) = \hbar\Delta_{Y0}^2 \frac{1+3m^2}{2\sqrt{2}}p_N(0)\sin\Omega t - \hbar\Delta_{Y0}^2\sqrt{3}mp_M(0)\sin\Omega t + \hbar\Delta_{Y0}^2p_Y(0)\sin\Omega t + q_Y(0)\cos\Omega t,$$

$$p'_+(t) = \frac{3(1-m^2) + (1+3m^2)\cos\Omega t}{4\sqrt{3}}p_N(0) + \frac{1+m-m\cos\Omega t}{\sqrt{2}}p_M(0) + \frac{\cos\Omega t}{\sqrt{6}}p_Y(0) - \frac{\sin\Omega t}{\sqrt{6}\hbar\Delta_{Y0}^2}q_Y(0),$$

$$p'_0(t) = \frac{3(1+m^2) - (1+3m^2)\cos\Omega t}{2\sqrt{3}}p_N(0) - \sqrt{2}m(1-\cos\Omega t)p_M(0) - \frac{2\cos\Omega t}{\sqrt{6}}p_Y(0) + \frac{2\sin\Omega t}{\sqrt{6}\hbar\Delta_{Y0}^2}q_Y(0),$$

$$p'_-(t) = \frac{3(1-m^2) + (1+3m^2)\cos\Omega t}{4\sqrt{3}}p_N(0) - \frac{1-m+m\cos\Omega t}{\sqrt{2}}p_M(0) + \frac{\cos\Omega t}{\sqrt{6}}p_Y(0) - \frac{\sin\Omega t}{\sqrt{6}\hbar\Delta_{Y0}^2}q_Y(0),$$

$$q'_+(t) = \left[\sqrt{3}Nt \left(\frac{(1-m)\varepsilon}{1+m} + 4(c_0+c_2)\mathcal{O}_{\phi\theta} \right) + \frac{\sqrt{3}\hbar\Delta_{Y0}^2}{8}(1+3m^2)(1+m)^2\sin\Omega t \right] p_N(0) \\ + \left[\frac{2\sqrt{2}\varepsilon}{1+m}Nt - \frac{3m(1-m)^2}{2\sqrt{2}}\hbar\Delta_{Y0}^2\sin\Omega t \right] p_M(0) + \hbar\Delta_{Y0}^2 \frac{3(1-m)^2\sin\Omega t}{2\sqrt{6}}p_Y(0) \\ + \frac{1}{\sqrt{3}}q_N(0) + \frac{1}{\sqrt{2}}q_M(0) + \left[-\frac{1-6m+3m^2}{2\sqrt{6}} + \frac{(1-m)^2}{2\sqrt{2}}\cos\Omega t \right] q_Y(0),$$

$$q'_0(t) = \left[\sqrt{3}Nt \left(\frac{1+m^2}{1-m^2}\varepsilon + 4(c_0+c_2)\mathcal{O}_{\phi\theta} \right) + \frac{\sqrt{3}}{8}\hbar\Delta_{Y0}^2(3m^4-2m^2-1)\sin\Omega t \right] p_N(0) + \left[-\frac{2\sqrt{2}m\varepsilon}{1-m^2}Nt \right. \\ \left. + \hbar\Delta_{Y0}^2 \frac{3m(1-m^2)}{2\sqrt{2}}\sin\Omega t \right] p_M(0) + \hbar\Delta_{Y0}^2 \frac{3m^2-3}{2\sqrt{6}}\sin\Omega t p_Y(0) + \frac{1}{\sqrt{3}}q_N(0) + \left[-\frac{1+3m^2}{2\sqrt{6}} + \frac{3m^2-3}{2\sqrt{6}}\cos\Omega t \right] q_Y(0),$$

$$\begin{aligned}
q'_-(t) = & \left[\sqrt{3}Nt \left(\frac{(1+m)\varepsilon}{1-m} + 4(c_0+c_2)\mathcal{O}_{\phi\theta} \right) + \frac{\sqrt{3}\hbar\Delta_{Y0}^2}{8}(1+3m^2)(1+m^2)\sin\Omega t \right] p_N(0) \\
& - \left[\frac{2\sqrt{2}\varepsilon}{1-m}Nt + \frac{3m(1+m)^2}{2\sqrt{2}}\hbar\Delta_{Y0}^2\sin\Omega t \right] p_M(0) + \hbar\Delta_{Y0}^2 \frac{3(1+m)^2\sin\Omega t}{2\sqrt{6}} p_Y(0) \\
& + \frac{1}{\sqrt{3}}q_N(0) - \frac{1}{\sqrt{2}}q_M(0) + \left[-\frac{1+6m+3m^2}{2\sqrt{6}} + \frac{(1+m)^2}{2\sqrt{2}}\cos\Omega t \right] q_Y(0).
\end{aligned} \tag{E1}$$

APPENDIX F: THE VARIANCE OF MACROSCOPIC CONDENSATE SPIN IN CARTESIAN COORDINATES

In Cartesian coordinates, the variances of the condensate spin direction is given by

$$\begin{aligned}
\langle [\delta f_x(t)]^2 \rangle &= \frac{1}{1-m^2} [3\langle p_N^2(0) \rangle - 2\sqrt{6}m\langle p_N(0)p_M(0) \rangle + 2m^2\langle p_M^2(0) \rangle], \\
\langle [\delta f_y(t)]^2 \rangle &= \frac{1-m^2}{2\hbar^2} \left[\langle q_M^2(0) \rangle + 2\sqrt{3}m\langle q_M(0)q_Y(0) \rangle + 3m^2\langle q_Y^2(0) \rangle \right] + \frac{4N^2\varepsilon^2t^2}{\hbar^2(1-m^2)} [3m^2\langle p_N^2(0) \rangle - 2\sqrt{6}m\langle p_N(0)p_M(0) \rangle \\
&+ 2\langle p_M^2(0) \rangle] + \frac{\sqrt{2}N\varepsilon t}{\hbar^2} \{ \sqrt{2}[\langle q_M(0)p_M(0) \rangle + \langle p_M(0)q_M(0) \rangle] - \sqrt{3}m[\langle q_M(0)p_N(0) \rangle + \langle p_N(0)q_M(0) \rangle] \\
&+ \sqrt{6}m[\langle q_Y(0)p_N(0) \rangle + \langle p_N(0)q_Y(0) \rangle] - 3m^2[\langle q_Y(0)p_M(0) \rangle + \langle p_M(0)q_Y(0) \rangle] \}, \\
\langle [\delta f_z(t)]^2 \rangle &= 2\langle p_M^2(0) \rangle.
\end{aligned} \tag{F1}$$

APPENDIX G: COEFFICIENTS IN THE ZERO MODE HAMILTONIAN FOR ANTIFERROMAGNETIC INTERACTIONS

The coefficients of zero mode Hamiltonian (77) for antiferromagnetic interactions (with $\mathcal{M}=0$)

$$\begin{aligned}
a &= \frac{(1+3n_0)\varepsilon}{6n_0(1-n_0)} + \left[6c_0 - \frac{8c_2n_0}{3(1-n_0)} \right] \mathcal{O}_{\phi\theta}, \quad b = \frac{\varepsilon}{1-n_0} + 4c_2\mathcal{O}_{\phi\theta}, \quad c = \frac{\varepsilon}{3n_0(1-n_0)} + \frac{4c_2n_0}{3(1-n_0)}\mathcal{O}_{\phi\theta}, \\
\alpha &= \frac{2(3n_0-1)\varepsilon}{3\sqrt{2}n_0(1-n_0)} + \frac{16c_2n_0}{3\sqrt{2}(1-n_0)}\mathcal{O}_{\phi\theta}, \quad \eta = 3c_2n_0(1-n_0)\mathcal{O}_{\phi\phi}/\hbar^2.
\end{aligned} \tag{G1}$$

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