Squashing minimum coverings of 6-cycles into minimum coverings of triples

Elizabeth J. Billington, Selda Küçükçifçi, C.C.
Lindner and Mariusz Meszka

Abstract. A 6-cycle is said to be squashed if we identify a pair of opposite vertices and name one of them with the other (thereby turning the 6-cycle into a pair of triples with a common vertex). A 6-cycle can be squashed in six different ways. The spectrum for 6-cycle systems that can be squashed into Steiner triple systems has been determined by Lindner, Meszka and Rosa [From squashed 6-cycles to Steiner triple systems, *J. Combin. Des.* [6]]. The squashing of maximum packings of K_n with 6-cycles into maximum packings of K_n with triples has also been fully dealt with by Lindner, Lo Faro and Tripodi [Squashing maximum packings of 6-cycles into maximum packings of triples, submitted]. The object of this paper is to extend these results to minimum coverings of K_n with 6-cycles into minimum coverings of K_n with triples. We give a complete solution.

Mathematics Subject Classification (2010). Primary 05C38; Secondary 05B30, 05B40 .

Keywords. Steiner triple systems; Squashed 6-cycle systems; Minimum coverings.

1. Introduction

Recently three papers by Lindner et al. have dealt with so-called “squashing” of 6-cycles into triples ([6], [4]) and 8-cycles into 4-cycles ([5]). This paper completes the problem regarding the existence for all orders of squashable 6-cycle systems, by considering *coverings*.

So we begin with some definitions.

A *Steiner triple system* $\text{STS}(n)$ of order n (also called simply a “triple system”) is a pair (X, T) where X is the vertex set of the complete graph K_n

and T is a collection of edge-disjoint triangles (also called “triples”) which partitions the edges of K_n .

A 6-cycle system of order n is a pair (X, C) where X is the vertex set of K_n and C is a collection of 6-cycles which partitions the edge-set of K_n .

One connection between 6-cycle systems and STS is given by 2-perfect 6-cycle systems; we omit details here, and refer interested readers to [7], or to the introduction in [6].

Here the connection between 6-cycles and triples is given by “squashing” a 6-cycle, which means identifying one pair of opposite vertices in the 6-cycle, forming a bowtie (that is, a pair of 3-cycles which share one vertex) from the 6-cycle. Each 6-cycle has three pairs of opposite vertices, and for each pair of opposite vertices there are two choices of vertex to remove in the “squashing”; so there are six possible bowties which can be formed from any one 6-cycle by squashing. See Figure 1, which illustrates this for the 6-cycle (a, b, c, d, e, f) .

If B is any one of the six bowties in Figure 1, we say that we have *squashed* (a, b, c, d, e, f) into B .

We note that the existence of bowtie systems for any admissible order was first determined by Horák and Rosa [2].

It is well-known that the spectrum (that is, the set of admissible orders) for 6-cycle systems is all positive integers n with $n \equiv 1$ or $9 \pmod{12}$ (see [8]), while the spectrum for $\text{STS}(n)$ is the set of all positive integers n with $n \equiv 1$ or $3 \pmod{6}$ ([3]). Thus the spectrum for which *both* a 6-cycle system *and* an $\text{STS}(n)$ exist is the set of all positive integers n with $n \equiv 1$ or $9 \pmod{12}$.

FIGURE 1

Let K_n have vertex set X . Then a *maximum packing* (X, C, L) of K_n with 6-cycles is a decomposition of $E(K_n) \setminus L$ (the edges of K_n with the edges

in L removed) into a maximum collection of cycles C , where the unused edges L form a set as small as possible. Thus for instance if $n \equiv 1$ or $9 \pmod{12}$, then $L = \emptyset$. The set L is called the *leave*.

In [6], a 6-cycle system of every order $n \equiv 1$ or $9 \pmod{12}$ which can be squashed into an STS(n) was constructed. That paper also constructs a 6-cycle maximum packing of order $n \equiv 3$ or $7 \pmod{12}$ which can be squashed into an STS(n). Subsequently, in [4], a 6-cycle maximum packing for every order n (regardless of the congruency class of n modulo 12), was constructed, which can be squashed into a maximum packing of K_n with triples. The interested reader is referred to [4] for details. The object of this paper is the obvious problem of extending this result to minimum coverings of K_n with 6-cycles. That is, we shall construct, for every possible order n with $n > 7$ a minimum covering of K_n with 6-cycles which can be squashed into a minimum covering of K_n with triples.

We first need some more definitions.

Let $P \subseteq E(K_n)$ and let C be a partition of $E(K_n) \cup P$ (where the union is strong, retaining repeated edges) into 6-cycles (or triples). If $|P|$ is as small as possible, then if the vertex set of K_n is X , we say that (X, C, P) is a *minimum covering* of K_n with 6-cycles (or triples). Here P is called the *padding*. (Several other papers have also used the term “excess” instead of “padding”; here we use the term padding.)

Table 1 lists some paddings for minimum coverings of K_n with 6-cycles, and also with triples. In some cases, for 6-cycles, other paddings might be possible, on the same number of edges as the paddings given below; however, we use the simple paddings given in the table.

Order n	6-cycles, padding	triples, padding
$n \equiv 0 \pmod{6}$	1-factor	1-factor
$n \equiv 2, 4 \pmod{6}$	$T + \triangle$	T
$n \equiv 1, 9 \pmod{12}$	$\emptyset$	$\emptyset$
$n \equiv 3, 7 \pmod{12}$	C_3	$\emptyset$
$n \equiv 5 \pmod{12}$	$C_5 \cup C_3$	C_5
$n \equiv 11 \pmod{12}$	C_5	C_5

Table 1: The paddings for 6-cycle coverings and for triple coverings of K_n

In Table 1, T denotes a tripole, that is, a spanning subgraph of odd degree with precisely one vertex of degree 3, and the rest of degree 1. Also $T + \Delta$ denotes a spanning subgraph of odd degree with precisely four vertices of degree 3 (forming a K_4) and the rest of degree 1. The padding denoted as $C_5 \cup C_3$ is here taken to be two disjoint cycles of lengths 5 and 3; however, other paddings are possible.

Now let (X, C, P) be a minimum covering of K_n with 6-cycles, where P is one of the paddings given in Table 1. Let B be a collection of bowties obtained by squashing the 6-cycles in C . If P contains no triples (when $n \equiv 0, 1, 6, 9$ or $11 \pmod{12}$), and (X, B, P) is a minimum covering of K_n with triples, or if P contains a triple b (when $n \equiv 2, 3, 4, 5, 7, 8$ or $10 \pmod{12}$), with $b \in B$ such that $(X, B \setminus b, P \setminus b)$ is a minimum covering of K_n with triples, then we will say that we have *squashed* (X, C, P) into a *minimum covering* of K_n with triples.

We illustrate this definition with two examples.

Example 1.1. A minimum covering of K_{11} with 6-cycles which squashes into a minimum covering of K_{11} with triples.

$C = \{(2, 3, 4, 5, 0, 1), (1, 2, 3, 4, 6, 0), (5, 3, 0, 2, 4, 1), (7, 1, 8, 9, 0, 4), (9, 2, 6, 7, 0, 4), (8, 2, 5, 7, 10, 0), (4, 8, 6, 1, 3, 10), (9, 3, 8, 5, 10, 1), (7, 3, 6, 9, 10, 2), (5, 6, 10, 8, 7, 9)\}$. The padding P is the 5-cycle $(0, 1, 2, 3, 4)$. $\square$

Here the padding contains no triples and

$B = \{(2, 3, 4), (2, 0, 1), (1, 2, 3), (1, 6, 0), (5, 3, 0), (5, 4, 1), (7, 1, 8), (7, 0, 4), (9, 2, 6), (9, 0, 4), (8, 2, 5), (8, 10, 0), (4, 8, 6), (4, 3, 10), (9, 3, 8), (9, 10, 1), (7, 3, 6), (7, 10, 2), (5, 6, 10), (5, 7, 9)\}$.

It is straightforward to see that (X, B, P) is a minimum covering of K_{11} with triples.

Example 1.2. A minimum covering of K_8 with 6-cycles which squashes into a minimum covering of K_8 with triples.

$C = \{(2, 3, 4, 5, 0, 1), (3, 6, 5, 2, 0, 1), (3, 0, 2, 4, 7, 1), (6, 7, 0, 3, 2, 1), (5, 7, 2, 6, 0, 4), (4, 1, 5, 3, 7, 6)\}$. The padding P is K_4 on $\{0, 1, 2, 3\}$ together with the edges $\{4, 5\}, \{6, 7\}$. $\square$

The padding here contains the triple $b = (0, 1, 2)$ which also belongs to B . Note that $(X, B \setminus \{(0, 1, 2)\}, P \setminus \{(0, 1, 2)\})$ is a minimum covering of K_8 with triples. Also $(P \setminus \{(0, 1, 2)\})$ is the tripole (see explanation below Table 1) with edges $\{\{3, 0\}, \{3, 1\}, \{3, 2\}, \{4, 5\}, \{6, 7\}\}$.

2. Preliminaries

From now on, to say that the 6-cycle (a, b, c, d, e, f) is squashed we will always refer to the *order* in which the cycle is written, with first vertex a , and mean that it has been squashed into the bowtie $(a, b, c)(a, e, f)$; see Figure 2.

FIGURE 2

So, for example, in Example 1.1 above, we can simply list the 6-cycles *in the given order* (without listing the bowties they have been squashed into) and state that they can be squashed into a triple system. The triples can be deduced from the given 6-cycles with the stated 6-cycle orders.

The following examples are used frequently in the rest of the paper.

Example 2.1. A minimum covering of $K_8 \setminus K_4$ with 6-cycles, which squashes into a minimum covering with triples.

The vertex set is $\{1, 2, 3, 4, 5, 6, 7, 8\}$; the padding is the two edges 12, 34, and the hole of order 4 is $\{5, 6, 7, 8\}$. The 6-cycle covering which can be squashed into triples is:

$$(1, 5, 2, 3, 4, 8), (1, 6, 2, 4, 3, 7), (3, 5, 4, 1, 2, 8), (4, 6, 3, 1, 2, 7).$$

□

Example 2.2. A minimum covering of $K_{10} \setminus K_4$ with 6-cycles, which squashes into a minimum covering with triples.

With vertex set $V(K_{10}) = \mathbb{Z}_{10}$, and hole $\{6, 7, 8, 9\}$, the following 6-cycles squash into triples; the padding consists of the three disjoint edges $\{0, 1\}, \{2, 3\}, \{4, 5\}$, which do not meet the hole.

$$(6, 1, 0, 7, 3, 2), (2, 1, 0, 6, 4, 7), (3, 5, 4, 2, 0, 8), (4, 5, 6, 3, 0, 9), (7, 1, 3, 4, 0, 5), \\ (1, 4, 8, 2, 9, 5), (2, 3, 9, 1, 8, 5).$$

□

Subsequently we shall make frequent use of *group divisible designs* (GDDs) in our constructions. A 3-GDD of type m^x consists of mx points partitioned into x groups of size m , with blocks (subsets) of size 3 satisfying the properties that every pair of points in different groups occurs in exactly one block, while pairs of points in the same group occur in no block. Similarly a 3-GDD of type $t^1 m^x$ consists of $t + mx$ points partitioned into one group of size t and x groups of size m , with blocks of size 3 satisfying the same requirements regarding pairs of points and groups. See [1], Section IV.4.1 for results on existence of such GDDs.

3. Even orders

3.1. Order 0 (mod 6)

It is not possible to take a minimum covering of K_6 with 6-cycles (that is, three 6-cycles and a 1-factor padding) which can be squashed into a 3-cycle minimum covering. This is shown in the following lemma.

Lemma 3.1. *There is no minimum covering of K_6 with 6-cycles which squashes into a minimum covering with triples.*

Proof. Consider first a decomposition of $K_6 \cup P$, where P is a matching of K_6 with three edges, into three bowties B_1 , B_2 and B_3 . We let $V(K_6) = \{1, 2, 3, 4, 5, 6\}$.

Each vertex in $K_6 \cup P$ has degree 6, so if a vertex has degree 4 in one bowtie, it must have degree 2 in a second bowtie, and not occur at all in the third bowtie. So without loss of generality we assume that vertex 1 has degree 4 in B_1 and degree 2 in B_2 ; we assume that vertex 2 has degree 4 in B_2 and degree 2 in B_3 ; and we assume that vertex 3 has degree 4 in B_3 and degree 2 in B_1 . Therefore edge $\{1, 3\}$ is in B_1 , edge $\{1, 2\}$ is in B_2 and edge $\{2, 3\}$ is in B_3 . Since vertex 1 does not occur in B_3 , vertex 2 does not occur in B_1 and vertex 3 does not occur in B_2 , it is not possible for any of the edges $\{1, 2\}$, $\{2, 3\}$, $\{3, 1\}$ to be in the matching P . So without loss of generality suppose that $P = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$. Therefore the edge $\{1, 4\}$ lies in both B_1 and B_2 , edge $\{2, 5\}$ lies in B_2 and B_3 , and edge $\{3, 6\}$ lies in B_3 and B_1 . We have now determined, without loss of generality, the three bowties:

$$\begin{aligned} B_1 &= \{\{1, 3, 6\}, \{1, 4, 5\}\}, \quad B_2 = \{\{2, 1, 4\}, \{2, 5, 6\}\}, \\ B_3 &= \{\{3, 2, 5\}, \{3, 4, 6\}\}. \end{aligned}$$

Now we need to transform each bowtie B_i , $i = 1, 2, 3$, into a 6-cycle C_i , $i = 1, 2, 3$, to give a decomposition of $K_6 \cup P$. Note that the edge $\{1, 2\}$ may appear in either C_2 or C_3 (in the latter case vertex 1 is the vertex “removed” when the 6-cycle is pinched).

Suppose first that $\{1, 2\} \in C_2$. Then cycle C_2 contains the path 2-1-4-3, while C_3 contains the path 3-2-5-1. Thus C_1 must contain the edges $\{1, 3\}$ and $\{2, 4\}$, so C_1 is determined: $C_1 = (1, 3, 6, 2, 4, 5)$. But then the edge $\{1, 5\}$ is in both C_1 and C_3 , a contradiction, since we assumed that $\{1, 5\} \notin P$.

The argument is similar if we assume instead that $\{1, 2\} \in C_3$: we find that 2-4-1-3 is a path in C_2 and 1-2-5-3 in C_3 , so that edges $\{2, 3\}$ and $\{1, 5\}$ must be in C_1 , so $C_1 = (2, 3, 6, 1, 5, 4)$ is forced. But now the edge $\{2, 4\}$ is in both C_1 and C_2 , a contradiction since $\{2, 4\} \notin P$.

Therefore no 6-covering of K_6 is possible which can be squashed into a 3-covering. $\square$

Next we give some necessary small examples.

Example 3.2. A minimum covering of K_{12} with 6-cycles which squashes into a minimum covering with triples.

We take $V(K_{12}) = A \cup B \cup C$, where $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7, 8\}$ and $C = \{9, 10, 11, 12\}$. Then we take three copies of Example 2.1 on the vertex sets $A \cup B$, $B \cup C$ and $C \cup A$, having holes (respectively) B , C and A . The result is a suitable minimum covering of K_{12} with 6-cycles which can be squashed into triples. $\square$

Example 3.3. A minimum covering of $K_{12} \setminus K_6$ with 6-cycles which squashes into a minimum covering with triples.

Let $V(K_{12}) = \mathbb{Z}_{12}$; the padding consists of the three edges $\{0, 1\}$, $\{2, 3\}$, $\{4, 5\}$ and the hole is $\{6, 7, 8, 9, 10, 11\}$; note that the padding does not meet the hole. The 6-cycles which can be squashed are:
 $(6, 1, 0, 7, 3, 2)$, $(7, 1, 0, 6, 3, 2)$, $(8, 2, 0, 9, 1, 4)$, $(9, 3, 0, 8, 1, 5)$, $(4, 0, 10, 3, 5, 6)$,
 $(5, 0, 11, 2, 4, 7)$, $(2, 1, 11, 5, 4, 9)$, $(3, 1, 10, 5, 4, 11)$, $(5, 2, 10, 4, 3, 8)$. $\square$

Example 3.4. A minimum covering of K_{18} with 6-cycles which squashes into a minimum covering with triples.

Let $V(K_{18}) = A \cup B \cup C$ where A, B, C are disjoint sets of size 6. Place a copy of Example 3.3 on the sets $A \cup B$, $B \cup C$ and $C \cup A$, where in each case the hole is the second listed set. The result is a suitable 6-cycle minimum covering which can be squashed into a 3-cycle minimum covering; the padding is a 1-factor of K_{18} . $\square$

Example 3.5. A minimum covering of K_{24} with 6-cycles which squashes into a minimum covering with triples.

We take $V(K_{24}) = \bigcup_{X=A,B,C} \{X_1 \cup X_2\}$ where each X_1, X_2 for $X = A, B, C$

is of order 4.

On the vertex set $X_1 \cup X_2$ (for each $X = A, B, C$) we place a copy of $K_8 \setminus K_4$ with hole X_1 . (See Example 2.1.) Then on the holes, $A_1 \cup B_1 \cup C_1$, we place a copy of K_{12} taken from Example 3.2. Finally, we use appropriate decompositions of $K_{4,4,4}$, three times, (see Example 9 in [6]) on each of the vertex sets $\{A_1 \cup B_2 \cup C_2\}$, $\{B_1 \cup A_2 \cup C_2\}$, $\{C_1 \cup A_2 \cup B_2\}$.

The result is a 6-cycle minimum covering of K_{24} which can be squashed into a minimum covering with triples; the padding is twelve disjoint edges (a 1-factor). $\square$

Example 3.6. A minimum covering of K_{30} with 6-cycles which squashes into a minimum covering with triples.

Let $V(K_{30}) = X \cup P \cup Q$ where $X = \{25, 26, \dots, 30\}$, $P = \{1, 2, 3, \dots, 12\}$ and $Q = \{13, 14, \dots, 24\}$. Now on the set P we can take a 4-GDD of type 3^4 , having four groups of size three, which we take as $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\{7, 8, 9\}$, $\{10, 11, 12\}$, and blocks of size 4. For each block $\{a, b, c, d\}$ in this 4-GDD, on the set $\{\{a, a + 12\}, \{b, b + 12\}, \{c, c + 12\}, \{d, d + 12\}\}$, we take a copy of

a 6-cycle *packing* of $K_{2,2,2,2}$ (which is the same as a 6-cycle packing of the graph K_8 minus a 1-factor), which squashes into a 3-cycle packing. Such an example is given in Example 2.2 of [4].

Then on the set $X \cup \{1, 2, 3, 13, 14, 15\}$ we place a copy of Example 3.2, for K_{12} . Next, on $X \cup \{4, 5, 6, 16, 17, 18\}$, on $X \cup \{7, 8, 9, 19, 20, 21\}$ and on $X \cup \{10, 11, 12, 22, 23, 24\}$, we place a copy of Example 3.3, for $K_{12} \setminus K_6$, each time taking the hole of size 6 to be the vertices in X .

The result is a 6-cycle minimum covering of K_{30} which can be squashed into a 3-cycle minimum covering. $\square$

We are now ready to deal with the general case.

Lemma 3.7. *Whenever $n \equiv 0 \pmod{6}$, $n > 6$, there is a minimum covering of K_n by 6-cycles which can be squashed into a minimum covering by triples.*

Proof. First, when the order n is 0 (mod 12), we let $n = 12x$ and assume $x \geq 3$; the cases $x = 1$ and $x = 2$ appear in Examples 3.2 and 3.5 above.

We take $V(K_{12x}) = \{v_1, v_2 \mid v \in \mathbb{Z}_{6x}\}$. On the set $\{v_1 \mid v \in \mathbb{Z}_{6x}\}$ we take a 3-GDD of type 6^x , since $x \geq 3$; see Section IV.4.1 of [1]. Then for each group of size 6, say $\{a_1, b_1, c_1, d_1, e_1, f_1\}$, on the set $\{a_1, b_1, c_1, d_1, e_1, f_1\} \cup \{a_2, b_2, c_2, d_2, e_2, f_2\}$ we place a copy of Example 3.2. Finally, for each block such as $\{p_1, q_1, r_1\}$ in the 3-GDD, on the set $\{p_1, p_2\} \cup \{q_1, q_2\} \cup \{r_1, r_2\}$ we place a copy of a 6-cycle decomposition of $K_{2,2,2}$ which squashes into a 3-cycle decomposition; see Example 2.1 in [4], because of course $K_{2,2,2}$ is the same graph as K_6 minus a 1-factor. This yields a minimum covering of K_{12x} by 6-cycles (with a 1-factor padding) which squashes into a minimum covering by triples.

Now consider K_{12x+6} where $x \geq 3$. The cases for orders 18 and 30, when $x = 1, 2$, appear in Examples 3.4 and 3.6 above.

We let $V(K_{12x+6}) = X \cup \{v_1, v_2 \mid v \in \mathbb{Z}_{6x+1}\}$, where $|X| = 4$ and $x \geq 3$. On the set $\{v_1 \mid v \in \mathbb{Z}_{6x+1}\}$ we take a 3-GDD of type $7^1 3^{2x-2}$; this exists provided $x \geq 3$ (see Section IV.4.1 of [1]). Suppose the one group of size 7 is $\{0_1, 1_1, 2_1, 3_1, 4_1, 5_1, 6_1\}$. Then we take minimum coverings with 6-cycles which squash into triples, as follows:

- (1) On the set $X \cup \{v_1, v_2 \mid 0 \leq v \leq 6\}$ we place a minimal covering of K_{18} with 6-cycles which can be squashed into a minimal covering by triples; see Example 3.4 above.
- (2) For each group $\{a_1, b_1, c_1\}$ of size 3 in the 3-GDD, on the set $X \cup \{a_i, b_i, c_i \mid i = 1, 2\}$ we place a minimal covering of $K_{10} \setminus K_4$ with 6-cycles which can be squashed into a minimal covering by triples, with the set X being the hole of size 4; see Example 2.2 above.
- (3) For each block $\{p_1, q_1, r_1\}$ in the 3-GDD, on the set $\{p_1, p_2\} \cup \{q_1, q_2\} \cup \{r_1, r_2\}$ we place a copy of a 6-cycle decomposition of $K_{2,2,2}$ which squashes into a 3-cycle decomposition; see Example 2.1 in [4].

The result is a minimum covering of K_{12x+6} by 6-cycles (with a 1-factor padding) which squashes into a minimum covering by triples. $\square$

3.2. Orders 2 and 4 (mod 6)

The case of order 8 is dealt with in Example 1.2 above. However, we also need the following examples here.

Example 3.8. A minimum covering of $K_8 \setminus K_2$ with 6-cycles which squashes into a minimum covering of K_8 with triples.

With vertex set $\mathbb{Z}_8$, let the hole be $\{6, 7\}$ and take the padding to be the disjoint edges (not meeting the hole): $\{0, 1\}, \{2, 3\}, \{4, 5\}$. Then squashable 6-cycles are as follows:

$$(2, 3, 4, 5, 0, 1), (1, 3, 6, 4, 7, 0), (5, 3, 0, 2, 4, 1), (4, 5, 7, 2, 6, 0), (2, 3, 7, 1, 6, 5).$$

□

Example 3.9. A minimum covering of K_{10} with 6-cycles which squashes into a minimum covering of K_{10} with triples.

$C = \{(2, 3, 4, 5, 0, 1), (2, 3, 5, 4, 0, 1), (5, 6, 0, 2, 4, 1), (4, 8, 0, 2, 5, 7), (3, 0, 9, 7, 6, 1), (7, 0, 3, 8, 9, 1), (8, 1, 3, 6, 9, 2), (6, 4, 9, 8, 7, 2), (8, 5, 9, 3, 7, 6)\}$. The padding P is K_4 on $\{0, 1, 2, 3\}$ together with the edges $\{4, 5\}, \{6, 7\}, \{8, 9\}$.

□

Example 3.10. A minimum covering of K_{14} with 6-cycles which squashes into a minimum covering of K_{14} with triples.

Let $V(K_{14}) = \mathbb{Z}_{14}$ and let $X = \{0, 1, 2, 3, 4, 5\}$ and $Y = \{6, 7, \dots, 13\}$. On the set Y we place a copy of a maximum *packing* of K_8 with 6-cycles which squashes into a maximum packing with triples, with leave the set of edges $\{6, 9\}, \{7, 8\}, \{10, 13\}, \{11, 12\}$; see Example 2.2 in [4] for this. Then on $X \setminus Y$, with padding K_4 on $\{0, 1, 2, 3\}$ together with the edges $\{4, 5\}, \{6, 7\}, \{8, 9\}, \{10, 11\}, \{12, 13\}$, we place the following 6-cycles which squash into triples:

$$\begin{aligned} (3, 2, 1, 0, 10, 5), & \quad (6, 5, 0, 1, 4, 3), & \quad (1, 2, 0, 3, 10, 11), & \quad (0, 2, 10, 1, 13, 3), \\ (0, 4, 12, 1, 3, 11), & \quad (1, 8, 4, 6, 0, 9), & \quad (7, 1, 6, 2, 8, 0), & \quad (1, 3, 12, 0, 13, 5), \\ (7, 2, 3, 9, 5, 4), & \quad (4, 10, 13, 12, 11, 2), & \quad (2, 5, 8, 7, 6, 9), & \quad (12, 5, 11, 4, 13, 2), \\ (9, 8, 3, 7, 5, 4). \end{aligned}$$

The result is a minimum covering of K_{14} with 6-cycles which squashes into a minimum covering of K_{14} with triples. □

Example 3.11. A minimum covering of K_{26} with 6-cycles which squashes into a minimum covering of K_{26} with triples.

Let $V(K_{26}) = \{\infty_1, \infty_2\} \cup \{x_1, x_2 \mid x \in \mathbb{Z}_{12}\}$. Take a 3-GDD on $\{x_1 \mid x \in \mathbb{Z}_{12}\}$ of type 3^4 . For one group, say $\{0_1, 1_1, 2_1\}$, on the set $\{\infty_1, \infty_2\} \cup \{x_1, x_2 \mid x = 0, 1, 2\}$, we place a copy from Example 1.2, of order 8, with the K_4 part of the padding being on the points $\{\infty_1, \infty_2, 0_1, 0_2\}$. Then on the remaining three groups of size 3 in the GDD, say $\{a_1, b_1, c_1\}$, on the set $\{\infty_1, \infty_2\} \cup \{x_1, x_2 \mid x = a, b, c\}$ we place a copy of Example 3.8, with the hole being $\{\infty_1, \infty_2\}$. Finally we use a decomposition of $K_{2,2,2}$ (as given in Example 2.1 in [4]) on the set $\{p_1, p_2\}, \{q_1, q_2\}, \{r_1, r_2\}$ for each block

$\{p_1, q_1, r_1\}$ in the GDD. This completes a minimum covering of K_{26} by 6-cycles which can be squashed into a minimum covering by triples. $\square$

Lemma 3.12. *Whenever $n \equiv 2$ or $4 \pmod{6}$, $n \geq 8$, there is a minimum covering of K_n by 6-cycles which can be squashed into a minimum covering by triples.*

Proof. We consider four cases according as the order is 2, 4, 8 or 10 (mod 12).

Order 4 (mod 12)

Let the vertex set of K_{12x+4} be $X \cup \{v_1, v_2 \mid v \in \mathbb{Z}_{6x}\}$, where $|X| = 4$. On the set $\{v_1 \mid v \in \mathbb{Z}_{6x}\}$ we take a 3-GDD of type 2^{3x} ; this exists for all $x \geq 1$ (see Section IV.4.1 of [1]). Then a 6-cycle minimum covering which can be squashed into a minimum covering of triples is obtained as follows:

- (1) For one group of the 3-GDD, say $\{0_1, 1_1\}$, on the set $X \cup \{0_1, 1_1, 0_2, 1_2\}$ we place a copy of Example 1.2; the padding is K_4 with two disjoint edges, and we make sure that the two disjoint edges do not belong to X , while the K_4 padding is on the set X .
- (2) For all other groups of the 3-GDD, say $\{a_1, b_1\}$, on the set $X \cup \{a_1, b_1, a_2, b_2\}$ we place a copy of Example 2.1, where X is the hole of size 4.
- (3) For each triple such as $\{p_1, q_1, r_1\}$ in the 3-GDD, on the set $\{p_1, p_2\} \cup \{q_1, q_2\} \cup \{r_1, r_2\}$ we place an appropriate decomposition of $K_{2,2,2}$, as given in Example 2.1 of [4].

The result is a suitable minimum covering of K_{12x+4} by 6-cycles which can be squashed into a minimum covering by triples; the padding is $T + \triangle$.

Order 8 (mod 12)

This case is exactly the same as the case 4 (mod 12), except that $\mathbb{Z}_{6x}$ is replaced by $\mathbb{Z}_{6x+2}$, and a 3-GDD of type 2^{3x+1} is used, which also exists for all $x \geq 1$. The case of order 8 (when $x = 0$) appears above in Example 1.2.

Order 10 (mod 12)

Consider K_{12x+10} . Note that when $x = 0$, the case for order 10 appears in Example 3.9 above. So we assume that $x \geq 1$. Let $V(K_{12x+10}) = X \cup \{v_1, v_2 \mid v \in \mathbb{Z}_{6x+3}\}$ where $|X| = 4$. On the set $\{v_1 \mid v \in \mathbb{Z}_{6x+3}\}$ we take a Kirkman Triple System. (In fact a 3-GGG of type 3^{2x+1} suffices.) Squashable minimum coverings by 6-cycles are then taken as follows:

- (1) If one group of the GDD is $\{0_1, 1_1, 2_1\}$, on the set $X \cup \{0_1, 1_1, 2_1, 0_2, 1_2, 2_2\}$ of size 10 we take a copy of Example 3.9.
- (2) For all other groups of the GDD, such as, say $\{a_1, b_1, c_1\}$, on the set $X \cup \{a_1, b_1, c_1, a_2, b_2, c_2\}$ we place a copy of Example 2.2, ensuring that the hole is the set X of size 4.
- (3) For each triple such as $\{p_1, q_1, r_1\}$ in the 3-GDD, on the set $\{p_1, p_2\} \cup \{q_1, q_2\} \cup \{r_1, r_2\}$ we place an appropriate decomposition of $K_{2,2,2}$, as given in Example 2.1 of [4].

Since all the above 6-cycles are squashable into minimum coverings of triples, this completes the case $10 \pmod{12}$.

Order $2 \pmod{12}$

Consider K_{12x+2} . When $x = 1$, the order is 14, and this case appears in Example 3.10 above. The case for $x = 2$ (order 26) is dealt with above in Example 3.11. So henceforth we assume that $x \geq 3$. We take $V(K_{12x+2}) = X \cup \{v_1, v_2 \mid v \in \mathbb{Z}_{6x-1}\}$ where $|X| = 4$. On the set $\{v_1 \mid v \in \mathbb{Z}_{6x-1}\}$ we take a 3-GDD of type $5^1 3^{2x-2}$ which exists for $x \geq 3$. Squashable minimum coverings by 6-cycles are then taken as follows:

- (1) Suppose the group of size 5 in the 3-GDD is $\{0_1, 1_1, 2_1, 3_1, 4_1\}$. Then on the set $X \cup \{v_1, v_2 \mid v = 0, 1, 2, 3, 4\}$, we place a copy of Example 3.10.
- (2) We repeat step (2) (for all groups of size 3 in this 3-GDD) in the above case of order $10 \pmod{12}$.
- (3) We repeat step (3) in the above case of order $10 \pmod{12}$.

The above four cases (modulo 12) complete the lemma. $\square$

4. Odd orders

When the order of the complete graph is 1 or 9 $\pmod{12}$, there is an exact 6-cycle system which can be squashed into a Steiner triple system; see [6]. So in this section we deal with the remaining odd orders (modulo 12), for which the padding is non-empty.

4.1. Orders 3 and 7 $\pmod{12}$

An exhaustive computer search has shown that it is not possible to find a minimum covering of K_7 (with the padding being one triangle) with 6-cycles which can be squashed into an STS(7) once the triangle padding is removed.

We now give some necessary small examples.

Example 4.1. A minimum covering of $K_{3,3,3}$ plus a padding of one 3-cycle, by 6-cycles, which squashes into a minimum covering of $K_{3,3,3}$ with triples.

With vertex set $\{0, 1, 2\} \cup \{3, 4, 5\} \cup \{6, 7, 8\}$, and padding the 3-cycle $(0, 3, 6)$, the following 6-cycles give a covering which squashes into a 3-cycle covering.

$(6, 0, 3, 1, 4, 2), (3, 2, 7, 5, 8, 0), (7, 0, 4, 6, 5, 1), (5, 2, 8, 3, 6, 0), (1, 6, 3, 7, 4, 8).$

$\square$

Example 4.2. A minimum covering of K_{15} with 6-cycles which squashes into a minimum covering of K_{15} with triples; the padding is one 3-cycle.

With vertex set $\mathbb{Z}_{15}$ and padding the 3-cycle $(0, 1, 2)$, the following 6-cycles give a covering which squashes into an STS(15) upon removal of the triangle padding.

$$\begin{array}{llll} (1, 0, 3, 10, 5, 2), & (0, 1, 2, 13, 10, 6), & (4, 0, 2, 7, 9, 1), & (11, 5, 0, 2, 14, 12), \\ (7, 0, 9, 4, 2, 6), & (14, 0, 8, 12, 2, 9), & (12, 13, 0, 10, 1, 6), & (8, 2, 11, 0, 12, 9), \\ (14, 7, 3, 1, 5, 6), & (1, 8, 10, 14, 11, 7), & (13, 14, 1, 11, 8, 7), & (2, 10, 12, 1, 13, 3), \\ (4, 3, 11, 9, 5, 8), & (3, 5, 12, 4, 6, 8), & (4, 13, 6, 3, 12, 7), & (10, 9, 3, 14, 5, 7), \\ (9, 13, 5, 4, 11, 6), & (10, 4, 14, 8, 13, 11). \end{array}$$

□

Example 4.3. A minimum covering of K_{19} with 6-cycles which squashes into a minimum covering of K_{19} with triples; the padding is one 3-cycle.

We take the vertex set $\{i_j \mid 1 \leq i \leq 6, j = 1, 2, 3\} \cup \{\infty\}$.

- (1) On the three sets $\{i_j, (i+1)_j \mid j = 1, 2, 3\} \cup \{\infty\}$, for $i = 1, 3, 5$, place a maximum packing of K_7 with 6-cycles which can be squashed into triples, with leave $\{(i+1)_1, (i+1)_2, (i+1)_3\}$, in each case (so $i+1 = 2, 4, 6$). (See Example 4 in [6].) This yields nine 6-cycles.
- (2) On the set $\{2_j, 4_j, 6_j \mid j = 1, 2, 3\}$, of size 9, place a 6-cycle system of K_9 which can be squashed into triples (see Example 1 in [6]); this is six 6-cycles, squashed into 12 triples.
- (3) For (3), (4) and (5), note that one STS(7) on $\{1, 2, 3, 4, 5, 6, \infty\}$ is given by the triples $\{\infty 12, \infty 34, \infty 56, 246, 235, 136, 145\}$.

On the set $\{2_1, 2_2, 2_3\} \cup \{3_1, 3_2, 3_3\} \cup \{5_1, 5_2, 5_3\}$ place a copy of the example in Lemma 11 of [6] ($K_{3,3,3} \setminus K_3$), with hole $\{2_1, 3_1, 5_1\}$. This contains four 6-cycles, squashable into eight triples.

- (4) Repeat (3) for $\{1_1, 3_1, 6_1\}$ instead of $\{2_1, 3_1, 5_1\}$.
- (5) Repeat (3) for $\{1_1, 4_1, 5_1\}$ instead of $\{2_1, 3_1, 5_1\}$.
- (6) So far we have used all edges of K_{19} except for the triples $\{2_1, 3_1, 5_1\}$, $\{1_1, 3_1, 6_1\}$, $\{1_1, 4_1, 5_1\}$. We now add in the padding $\{2_1, 4_1, 6_1\}$. So we take a maximum packing of $K_{2,2,2}$ on the vertex set $\{1_1, 2_1\} \cup \{3_1, 4_1\} \cup \{5_1, 6_1\}$, with two 6-cycles which squash into triples:

$$(6_1, 1_1, 3_1, 5_1, 4_1, 2_1), (5_1, 2_1, 3_1, 6_1, 4_1, 1_1).$$

We now have K_{19} plus the padding $\{2_1, 4_1, 6_1\}$ decomposed into 29 6-cycles which can be squashed into 58 triples, one of which is $\{2_1, 4_1, 6_1\}$. Discarding this triple leaves an STS(19) on 57 triples, as required.

□

Example 4.4. A minimum covering of K_{27} with 6-cycles which squashes into a minimum covering of K_{27} with triples; the padding is one 3-cycle.

We take the vertex set $\{i_j \mid 1 \leq i \leq 9, j = 1, 2, 3\}$. Then 6-cycles (which squash into triples) are taken as follows:

- (1) On the sets $\{1_j, 2_j, 3_j \mid j = 1, 2, 3\}$, $\{4_j, 5_j, 6_j \mid j = 1, 2, 3\}$ and $\{7_j, 8_j, 9_j \mid j = 1, 2, 3\}$ we place squashable 6-cycle systems of order 9. (We also use a Kirkman triple system KTS(9) on $\{i \mid 1 \leq i \leq 9\}$, with one parallel class $\{1\,2\,3, 4\,5\,6, 7\,8\,9\}$.)
- (2) There are nine more triples in the KTS(9) besides $\{1\,2\,3, 4\,5\,6, 7\,8\,9\}$. For each of these nine triples, say $\{x\,y\,z\}$, on the set $\{x_j \mid j = 1, 2, 3\} \cup \{y_j \mid j = 1, 2, 3\} \cup \{z_j \mid j = 1, 2, 3\}$, we place a maximum packing of $K_{3,3,3}$ with 6-cycles, with leave $\{x_1, y_1, z_1\}$, which can be squashed into triples (see Lemma 11 of [6]).
- (3) The edges of K_{27} which are so far unused are those in $K_{3,3,3}$ on the vertex set $\{1_1, 2_1, 3_1\} \cup \{4_1, 5_1, 6_1\} \cup \{7_1, 8_1, 9_1\}$. We include the padding $\{1_1, 4_1, 7_1\}$, and place a copy of Example 4.1 ($K_{3,3,3}$ plus a triangle) on here.

This completes the case of order 27.

□

Example 4.5. A minimum covering of K_{31} with 6-cycles which squashes into a minimum covering of K_{31} with triples; the padding is one 3-cycle.

Let $V(K_{31}) = X \cup \{x_1, x_2 \mid x \in \mathbb{Z}_{14}\}$, where $|X| = 3$. There exists a 3-GDD of type $6^1 2^4$ on 14 points, so we use decompositions of K_{15} (Example 4.2), with the group of size 6, and $K_7 \setminus K_3$ (see Example 4 in [6]) with the four groups of size 2, and then $K_{2,2,2}$ as usual with each block of the GDD.

□

We can now deal with the general case when the order is 3 or 7 (mod 12).

Lemma 4.6. *Whenever $n \equiv 3$ or 7 (mod 12), $n \geq 15$, there is a minimum covering of K_n by 6-cycles which can be squashed into a minimum covering by triples.*

Proof. For K_{12x+3} , note that $12x+3 = 3 + 2(6 + 2(3x-3))$. Our construction requires $x \geq 3$ in order to use a 3-GDD of type $6^1 2^{3x-3}$. When $x = 1$, Example 4.2 provides a solution, and when $x = 2$, Example 4.4 provides a solution. For order $12x+3$ when $x \geq 3$, take the vertex set $\{i_j \mid 1 \leq i \leq 6x, j = 1, 2\} \cup \{\infty_1, \infty_2, \infty_3\}$. Let the groups of a 3-GDD of type $6^1 2^{3x-3}$ be $\{1, 2, 3, 4, 5, 6\}, \{7, 8\}, \dots, \{6x-1, 6x\}$. On the set $\{i_j \mid 1 \leq i \leq 6, j = 1, 2\} \cup \{\infty_1, \infty_2, \infty_3\}$ we place a copy of Example 4.2, with padding $\{\infty_1, \infty_2, \infty_3\}$. Then for each group of the GDD $\{a, a+1\}$ of size 2, on the set $\{a_1, a_2, (a+1)_1, (a+1)_2, \infty_1, \infty_2, \infty_3\}$ we place a squashable 6-cycle maximum packing of K_7 (Example 4 in [6]) with leave $\{\infty_1, \infty_2, \infty_3\}$. Finally for each block, say pqr of the 3-GDD, on the set $\{p_1, p_2\} \cup \{q_1, q_2\} \cup \{r_1, r_2\}$ we place a copy of a squashable 6-cycle decomposition of $K_{2,2,2}$ (see Example 2.1 in [4]).

Similarly, for K_{12x+7} , we note that $12x+7 = 3 + 2(8 + 2(3x-3))$, and we use a 3-GDD of type $8^1 2^{3x-3}$, which exists provided $x \geq 3$. When $x = 0$, order 7 cannot be done; for $x = 1$ (order 19), see Example 4.3; and for $x = 2$

(order 31), see Example 4.5. For the general case, with $x \geq 3$, as in the case $3 \pmod{12}$ we use Example 4.3 once (instead of order 15 as in the case $3 \pmod{12}$), and then the rest proceeds exactly as in the case $3 \pmod{12}$. $\square$

4.2. Order 5 (mod 12)

We begin with an example of order 9 having two disjoint holes of size 3.

Example 4.7. A decomposition of K_9 with two disjoint holes of size 3, into 6-cycles which squash into a triple decomposition.

We take the vertex set $\mathbb{Z}_9$, with the two disjoint holes $\{0, 1, 2\}$ and $\{3, 4, 5\}$. The squashable 6-cycles are as follows:

$$(4, 7, 1, 3, 0, 8), (3, 8, 1, 4, 0, 6), (5, 0, 7, 2, 6, 1), (2, 3, 7, 6, 5, 8), (6, 4, 2, 5, 7, 8).$$

$\square$

Example 4.8. A minimum covering of K_{17} with 6-cycles which squashes into a minimum covering of K_{17} with triples; the padding is a 5-cycle and a 3-cycle.

We take the vertex set $\mathbb{Z}_{17}$, and padding the 3-cycle $(0, 1, 2)$ and 5-cycle $(3, 4, 5, 6, 7)$. The following 6-cycles squash into triples, and leave a hole on the vertices $\{8, 9, 10, 11, 12, 13, 14, 15, 16\} \setminus \{(8, 9, 10), (12, 13, 14)\}$. By filling this hole with a system of order 9 having two disjoint holes of size 3 (see Example 4.7), we complete the example of order 17.

$$\begin{array}{llll} (0, 1, 2, 4, 6, 11), & (2, 12, 1, 0, 13, 3), & (8, 2, 0, 10, 9, 1), & (4, 11, 2, 0, 6, 10), \\ (0, 3, 15, 2, 5, 12), & (9, 4, 0, 8, 10, 2), & (2, 16, 5, 0, 15, 7), & (10, 7, 0, 9, 8, 5), \\ (0, 14, 13, 5, 1, 16), & (5, 4, 13, 2, 1, 14), & (7, 5, 4, 3, 1, 11), & (15, 1, 4, 16, 6, 5), \\ (13, 1, 6, 14, 7, 12), & (3, 8, 4, 7, 1, 10), & (14, 2, 6, 13, 7, 3), & (16, 3, 4, 12, 6, 7), \\ (12, 3, 6, 15, 4, 14), & (7, 8, 6, 5, 3, 9), & (5, 9, 6, 7, 3, 11). \end{array}$$

$\square$

Example 4.9. A minimum covering of $K_{5,5,5} \setminus K_3$ with 6-cycles which squashes into a triple decomposition minus one K_3 .

We take the vertex set $\{0, 1, 2, 3, 4\} \cup \{5, 6, 7, 8, 9\} \cup \{10, 11, 12, 13, 14\}$, and remove the triangle $\{0, 5, 10\}$. The following 6-cycles squash into a triple decomposition (missing the triple $\{0, 5, 10\}$).

$$\begin{array}{llll} (9, 4, 14, 8, 0, 13), & (8, 3, 14, 9, 0, 12), & (7, 0, 11, 9, 2, 14), & (6, 0, 14, 5, 4, 13), \\ (10, 1, 8, 13, 3, 6), & (9, 1, 12, 7, 2, 10), & (4, 10, 7, 1, 11, 8), & (1, 6, 11, 4, 7, 13), \\ (5, 1, 14, 6, 4, 12), & (13, 2, 8, 10, 3, 5), & (12, 6, 2, 11, 7, 3), & (11, 5, 2, 12, 9, 3). \end{array}$$

$\square$

Example 4.10. A minimum covering of $K_{17} \setminus K_5$ with 6-cycles which squashes into a minimum covering of $K_{17} \setminus K_5$ with triples.

We take the vertex set to be $\{i_j \mid 1 \leq i \leq 5, j = 1, 2, 3\} \cup \{\infty_1, \infty_2\}$. The hole of size 5 will be $\{1_1, 1_2, 1_3, \infty_1, \infty_2\}$. On the three sets $\{1_j, 2_j, 3_j, 4_j, 5_j, \infty_1, \infty_2\}$, for $j = 1, 2, 3$, we place a maximum packing of K_7 with 6-cycles which can be squashed into triples (see Example 4 in [6]) with leave $\{1_j, \infty_1, \infty_2\}$ for each j . Then on the sets $\{i_1 \mid i \leq 5\} \cup \{i_2 \mid i \leq 5\} \cup \{i_3 \mid i \leq 5\}$ we place a copy of Example 4.9 where the leave is the set $\{1_1, 1_2, 1_3\}$. This completes the example. $\square$

The following example is needed to complete the case of order 29.

Example 4.11. A covering of $K_{11} \setminus (K_5 \cup K_3)$, where the holes of size 5 and 3 share one vertex, with 6-cycles, which squashes into a collection of triples.

We take the vertex set $\mathbb{Z}_{11}$, where the holes of size 3 and 5 are $\{0, 1, 2\} \cup \{2, 3, 4, 5, 6\}$ (the holes share one common vertex).

The following 6-cycles squash into a triple decomposition:

$$\begin{aligned} (7, 5, 0, 3, 1, 4), & \quad (0, 4, 8, 2, 9, 6), & \quad (3, 9, 1, 7, 0, 10), & \quad (2, 7, 9, 0, 8, 10), \\ (1, 5, 10, 7, 8, 6), & \quad (9, 4, 10, 1, 8, 5), & \quad (7, 6, 10, 9, 8, 3). \end{aligned}$$

$\square$

Example 4.12. A minimum covering of K_{29} with 6-cycles which squashes into a minimum covering of K_{29} with triples; the padding is a 5-cycle and a 3-cycle.

We take the vertex set $\{i_j \mid 1 \leq i \leq 8, j = 1, 2, 3\} \cup X$, where $X = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$.

- (1) On the set $X \cup \{1_j, 2_j \mid j = 1, 2, 3\}$, we place a copy of Example 1.1, which has ten 6-cycles and padding a 5-cycle (which we ensure lies entirely within the set X).
- (2) On the three sets $X \cup \{3_j, 4_j \mid j = 1, 2, 3\}$, $X \cup \{5_j, 6_j \mid j = 1, 2, 3\}$, and $X \cup \{7_j, 8_j \mid j = 1, 2, 3\}$, we place a copy of Example 4.11, of K_{11} with holes of size 5 (on X) and size 3 (being $\{\infty_5, 3_1, 4_1\}$, $\{\infty_5, 5_1, 6_1\}$, $\{\infty_5, 7_1, 8_1\}$, respectively).
- (3) On the set $\{\infty_5, 1_1, 2_1, \dots, 7_1, 8_1\} = Y$, we take an STS(9) containing the triples $\{\infty_5, 1_1, 2_1\}$, $\{\infty_5, 3_1, 4_1\}$, $\{\infty_5, 5_1, 6_1\}$, $\{\infty_5, 7_1, 8_1\}$.
- (4) For each of the eight triples not involving ∞_5 in the STS(9) in (3), such as, say $\{x_1, y_1, z_1\}$, on the sets $\{x_1, x_2, x_3\} \cup \{y_1, y_2, y_3\} \cup \{z_1, z_2, z_3\}$ we place a decomposition of $K_{3,3,3} \setminus \{x_1, y_1, z_1\}$ into four squashable 6-cycles (see Lemma 11 in [6]).
- (5) All edges in K_{29} have now been covered except for those in $Y \setminus \{\infty_5, 1_1, 2_1\}$ (see (3)). We now adjoin the padding $\{\infty_5, 1_1, 2_1\}$. Then Y can be partitioned into six squashable 6-cycles (see Example 1 in [6]) so that $\{\infty_5, 1_1, 2_1\}$ is one of the triples from the squashing.

- (6) Now removing the triple $\{\infty_5, 1_1, 2_1\}$ gives a minimum covering of K_{29} with triples from the squashed 6-cycles, with padding the 5-cycles inside X .

□

Lemma 4.13. *Whenever $n \equiv 5 \pmod{12}$, $n \geq 17$, there is a minimum covering of K_n by 6-cycles which can be squashed into a minimum covering by triples.*

Proof. Order 17 is covered by Example 4.8, and order 29 by Example 4.12. So consider K_{12x+5} where $x \geq 3$. For order $12x + 5$, we take the vertex set

$$\{i_j \mid 1 \leq i \leq 6x, j = 1, 2\} \cup \{\infty_k \mid k = 1, 2, 3, 4, 5\}.$$

We use a 3-GDD of type 6^x (which exists for $x \geq 3$; see [1]) on the set $\{1, 2, \dots, 6x\}$, with groups $\{6a + 1, 6a + 2, \dots, 6a + 6\}$ for $a = 0, 1, \dots, x - 1$. Then we use Example 4.8 once, on the set $\{i_j \mid 1 \leq i \leq 6, j = 1, 2\} \cup \{\infty_k \mid 1 \leq k \leq 5\}$; Example 4.10, $x - 1$ times, on $\{(6a + i)_j \mid 1 \leq i \leq 6, j = 1, 2\} \cup \{\infty_k \mid 1 \leq k \leq 5\}$, for $a = 1, 2, \dots, x - 1$, with the obvious hole of size 5; and for each triple pqr of the 3-GDD, on the sets $\{p_1, p_2\} \cup \{q_1, q_2\} \cup \{r_1, r_2\}$, we use our squashable 6-cycle decomposition of $K_{2,2,2}$ (see Example 2.1 in [4]). □

4.3. Order 11 (mod 12)

Recall that the case of order 11 was dealt with earlier in Example 1.1. No further examples are required for orders $11 \pmod{12}$. We have the following result.

Lemma 4.14. *Whenever $n \equiv 11 \pmod{12}$, $n \geq 11$, there is a minimum covering of K_n by 6-cycles which can be squashed into a minimum covering by triples.*

Proof. The case of order 11 appears in Example 1.1, so we consider K_{12x+11} where $x > 0$. We let the vertex set be $X \cup \{v_1, v_2 \mid v \in \mathbb{Z}_{6x+4}\}$ where $|X| = 3$, and we use a 3-GDD of type $4^1 2^{3x}$, which exists for $x > 0$. Let the group of size 4 be $\{0_1, 1_1, 2_1, 3_1\}$. The “ingredients” are then (i) Example 1.1 on the set $X \cup \{0_1, 1_1, 2_1, 3_1, 0_2, 1_2, 2_2, 3_2\}$ (with its padding C_5 not meeting the set X); (ii) a maximum packing of order 7 with leave the set X on the sets $X \cup \{a_1, b_1, a_2, b_2\}$ for each group $\{a_1, b_1\}$ in the GDD; and (iii) copies of $K_{2,2,2}$, using each block of size 3 from the GDD in the usual way.

The result is a suitable minimum covering of K_{12x+11} by 6-cycles, with padding a 5-cycle, which can be squashed into a minimum covering by triples. □

5. Concluding comments

Combining results from [6] with the Lemmas 3.7, 3.12, 4.6, 4.13, and 4.14 above, we have now shown the following.

Theorem 5.1. *For each order $n \geq 8$, there is a minimum covering of K_n with 6-cycles which can be squashed into a minimum covering of K_n with triples.*

References

- [1] Colbourn, C.J., Dinitz, J.H. (eds.): The CRC Handbook of Combinatorial Designs. Second Edition, Chapman and Hall/CRC, Boca Raton, FL (2007).
- [2] Horák, P., Rosa, A.: Decomposing Steiner triple systems into small configurations. *Ars Combin.* 26, 91–105 (1988).
- [3] Kirkman, T.P.: On a problem in combinations. *Cambridge and Dublin Math. J.* 2, 191–204 (1847).
- [4] Lindner, C.C., Lo Faro, G., Tripodi, A.: Squashing maximum packings of 6-cycles into maximum packings of triples, (submitted).
- [5] Lindner, C.C., Lo Faro, G., Meszka, M., Tripodi, A.: Squashing maximum packings of K_n with 8-cycles into maximum packings of K_n with 4-cycles, *Filomat* (to appear).
- [6] Lindner, C.C., Meszka, M., Rosa, A.: From squashed 6-cycles to Steiner triple systems. *J. Combin. Des.* 22, 189–195 (2014).
- [7] Lindner, C.C., Phelps, K.T., Rodger, C.A.: The spectrum for 2-perfect 6-cycle systems. *J. Combin. Theory Ser. A* 57, 76–85 (1991).
- [8] Rosa, A. : On cycle decompositions of the complete graph into $(4m + 2)$ -gons. *Mat-fyz Časopis.* 16, 349–435 (1966).

Elizabeth J. Billington
 School of Mathematics and Physics
 The University of Queensland
 Brisbane, QLD 4072, Australia
 e-mail: ejb@maths.uq.edu.au

Selda Küçükçifçi
 Department of Mathematics, Koç University
 Rumelifeneri Yolu, 34450, Sarıyer
 Istanbul, Turkey
 e-mail: skucukcifci@ku.edu.tr

C.C. Lindner
 Department of Mathematics and Statistics
 Auburn University
 Auburn, AL 36849-5307, U.S.A.
 e-mail: lindncc@auburn.edu

Mariusz Meszka
 AGH University of Science and Technology
 Kraków, 30059 Poland
 e-mail: meszka@agh.edu.pl