Comment on “Quartic anharmonic oscillator and non-Hermiticity”

Ali Mostafazadeh

Department of Mathematics, Koç University, 34450 Sariyer, Istanbul, Turkey

(Received 31 August 2004; published 8 April 2005)

The analysis of the paper by J.-L. Chen *et al.* [Phys. Rev. A **67**, 012101 (2003)] suffers from a conceptual error. Its results contradict some well-established mathematical facts and are incorrect.

DOI: 10.1103/PhysRevA.71.046101

PACS number(s): 03.65.Fd, 34.20.Gj

In [1] the authors argue that the Hamiltonian

$$H = p^2 + V(x), \quad V(x) = \rho x + \alpha x^2 + \beta x^3 + \gamma x^4, \quad (1)$$

with γ real and positive, does not have an analytic (normalizable) eigenfunction, and that for γ real and negative the existence of an analytic (normalizable) eigenfunction of this Hamiltonian forces the potential to be nonreal. The aim of this Comment is to show that both of these statements are false.

It is a well-known mathematical theorem that for any $E \in \mathbb{R}$ the general solution of

$$\left[-\frac{d^2}{dx^2} + V(x) \right] \Psi(x) = E \Psi(x) \quad (2)$$

has the form $\Psi(x) = c_1 \Psi_1(x) + c_2 \Psi_2(x)$, where Ψ_1 and Ψ_2 are a pair of linearly independent (fundamental) solutions [2]. Furthermore, if V is an analytic function, so are Ψ_1 and Ψ_2 , and consequently any solution of (2) (see Theorem 5.3.1 of [3] and [2] for its proof.) Therefore, the claim that the Hamiltonian (1) can have nonanalytic eigenfunctions is unfounded. In fact, for $\gamma > 0$, this Hamiltonian has a nondegenerate discrete (pure point) spectrum [4,5] and all its (normalizable) eigenfunctions are analytic. The root of the mistaken statements made in [1] is a conceptual error that I briefly elude to below.

The analysis of [1] involves the following steps. First, (2) is restricted to the positive ($\mathbb{R}^+$) and then negative ($\mathbb{R}^-$) real axes. On $\mathbb{R}^+$, the following change of variable is made: $\Psi(x) \rightarrow \phi(x) = g(x)\Psi(x)$, where $g(x) = (\mathcal{N}_0^\pm)^{-1} e^{\int W_\pm(x) dx}$, $W_\pm(x) = \mu_\pm x^2 + \tau_\pm x + \nu_\pm$ and $\mu_\pm$, $\tau_\pm$, and $\nu_\pm$ are constants. The resulting equation for ϕ is an eigenvalue equation for an effective Hamiltonian H that can be expressed in terms of the generators of a realization of the algebra $\mathfrak{su}(2)$ that involves a numerical value j . In this realization the Casimir operator

$\mathbf{j}^2 := j_0^2 + \{j_+, j_-\}/2$ is given by $\mathbf{j}^2 = j(j+1)$. The authors of [1] use the fact that H commutes with $\mathbf{j}^2$ to conclude that H must have eigenfunctions with definite j values. In particular, they claim that the ground-state eigenfunction is the $j=0$ state.

In order to see the flaw in this argument, first note that the functions ϕ are not normalizable with respect to the ordinary L^2 norm. One can identify the transformation $\Psi \rightarrow \phi$ with a linear operator u mapping $L^2(\mathbb{R}^\pm)$ to the space of complex-valued functions as $(u\Psi)(x) := g(x)\Psi(x)$, and introduce the following inner product on the range $\mathcal{R}$ of u : $(\phi_1 | \phi_2) := \int_{\mathbb{R}^\pm} g(x)^{-2} \phi_1(x)^* \phi_2(x) dx$. Then by construction $u: L^2(\mathbb{R}^\pm) \rightarrow \mathcal{R}$ is a unitary operator, i.e., for all $\Psi_1, \Psi_2 \in L^2(\mathbb{R})$, $\langle \Psi_1 | \Psi_2 \rangle = (u\Psi_1 | u\Psi_2)$, where $\langle \cdot | \cdot \rangle$ is the usual L^2 inner product. The eigenvalue problem for the effective Hamiltonian H must be solved in the space $\mathcal{R}$ endowed with the inner product $(\cdot | \cdot)$, and the generators $j_1 = (j_+ + j_-)/2$, $j_2 = (j_+ - j_-)/(2i)$, and $j_3 = j_0$ must be viewed as acting in $\mathcal{R}$. However, unlike H , these generators are not Hermitian operators acting in $\mathcal{R}$. Therefore, the above-mentioned realization of $\mathfrak{su}(2)$ is not a unitary representation. Furthermore, because the lowering operator j_- is independent of j , the vacuum state ϕ_0 defined by $j_- \phi_0 = 0$, which corresponds to the constant function $\phi_0(x) = 1$, is the same for all the values of j . This shows that $\mathcal{R}$ does not admit an orthogonal direct sum decomposition into subspaces that are associated with the above-mentioned j realizations of $\mathfrak{su}(2)$ with different values of j . In particular, one cannot use the usual construction of the eigenfunctions of the Hamiltonian by restricting to these definite j subspaces as done in [1]. Indeed, if this was possible, all the eigenfunctions of the initial Hamiltonian (1) would have the general form $g^{-1}(x)P(x)$, where $P(x)$ is a polynomial, but one can easily check that this is not the case.

In summary, the method used in [1] to study the eigenfunctions of the Hamiltonian (1) is inapplicable and the results reported in this paper are erroneous.

[1] J.-L. Chen, L. C. Kwek, and C. H. Oh, Phys. Rev. A **67**, 012101 (2003).

[2] E. Ince, *Ordinary Differential Equations* (Dover, New York, 1956).

[3] W. E. Boyce and R. C. DiPrima, *Elementary Differential*

Equations and Boundary Value Problems (Wiley, New York, 2001).

[4] A. Messiah, *Quantum Mechanics* (Dover, New York, 1999).

[5] E. Hille, *Lectures on Ordinary Differential Equations* (Addison-Wesley, Reading, MA, 1969).