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2016 EPL 114 60004

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New improved massive gravity

T. DERELI and C. YETİŞMİŞOĞLU

Department of Physics, Koç University - 34450 Sariyer, Istanbul, Turkey

received 15 April 2016; accepted in final form 20 June 2016
published online 15 July 2016

PACS 04.60.Rt – Topologically massive gravity
PACS 04.60.Kz – Lower dimensional models; mini-superspace models
PACS 04.20.Fy – Canonical formalism, Lagrangians, and variational principles

Abstract – We derive the field equations for topologically massive gravity coupled with the most general quadratic curvature terms using the language of exterior differential forms and a first-order constrained variational principle. We find variational field equations both in the presence and absence of torsion. We then show that spaces of constant negative curvature (*i.e.* the anti de-Sitter space AdS_3) and constant torsion provide exact solutions.

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The well-known topologically massive gravity (TMG) theory in 3 dimensions [1,2] provides deep insight into problems related with the topology of gravitational fields and suggests valuable answers to basic questions such as can gravitons have a finite range, or what would be the nature of quantum gravity. In this theory, a gravitational degree of freedom that propagates in free space is introduced by raising the second-order Einstein field equations with a cosmological constant to third-order by adding the gravitational Chern-Simons term to the action. In particular, TMG admits stationary, circularly symmetric solutions that are asymptotically AdS_3 and share many common features with the unique rotating black-hole solution in 4 dimensions [3,4]. A remarkable, unitary extension of TMG has been given recently under the name of new massive gravity (NMG) [5]. Here the Chern-Simons term in the action is dropped, but instead a certain combination of quadratic curvature invariants that implies unitarity are introduced. Therefore the corresponding variational field equations do not involve third-order derivatives but fourth order [6]. In both NMG and TMG (and its quadratic curvature extensions that are considered so far), the space-time geometry is (pseudo)-Riemannian. There is a more recent extension of Einstein-Chern-Simons gravity, called the minimal massive 3D gravity (MMG), that involves a dynamical space-time torsion [7]. This is achieved by imposing an unconventional constraint on the torsion 2-forms that involves terms that are non-linear in Lagrange multipliers. Here in this short note, we consider the field equations of Einstein-Chern-Simons gravity with a cosmological constant that is coupled with the most general quadratic curvature terms in the action. Field equations

both in the presence and absence of torsion are derived by a first-order constrained variational principle. The algebra of exterior differential forms is employed in an essential way [8]. The notion of the torsion of a material continuum has been first introduced by Cartan in 1922 [9]. This idea found important physical applications, on the one hand, within the context of modified theories of gravity [10], while on the other hand, within the context of gauge theories of continuum dislocation and disclination defects [11,12]. Here we will be discussing 3-dimensional space-times of constant negative curvature (*i.e.* AdS_3) and constant torsion as exact background solutions. The notion of constant torsion in 3 dimensions [13] was implicit in Cartan's work and it is called Cartan's spiral staircase in a recent review [11]. Details of the calculations that follow were presented elsewhere [14].

The field equations of our model will be determined by the constrained variations of an action integral

$$I[e^a, \omega_b^a, \lambda_a] = \int_M \mathcal{L}, \quad (1)$$

where M is a compact region on some chart on a (1+2)-dimensional Riemann-Cartan manifold. The independent variables on which the action depends are the co-frame 1-forms $\{e^a\}$, connection 1-forms $\{\omega_b^a\}$, and Lagrange multiplier 1-forms $\{\lambda_a\}$. In terms of these basic variables, the metric tensor is given by $g = \eta_{ab} e^a \otimes e^b$, where $\eta_{ab} = \text{diag}(- + +)$. The spacetime orientation is set with the choice of the volume 3-form $*1 = e^0 \wedge e^1 \wedge e^2 = 1/3! \epsilon_{abc} e^{abc}$ where $*$ denotes the Hodge map that sends p -forms to $(3 - p)$ -forms and we use a shorthand for the products $e^a \wedge e^b \wedge \dots = e^{ab\dots}$. The basis 1-forms together with

the metric-compatible connection 1-forms ($\omega_{ba} = -\omega_{ab}$) satisfy the Cartan structure equations

$$\begin{aligned} De^a &\equiv de^a + \omega^a_b \wedge e^b = T^a, \\ D\omega^a_b &\equiv d\omega^a_b + \omega^a_c \wedge \omega^c_b = R^a_b. \end{aligned} \quad (2)$$

Contractions of the curvature 2-forms R^a_b yield the Ricci 1-forms $Ric_a = \mathcal{R}_{ab}e^b$ and the curvature scalar $\mathcal{R}$ in the usual way. It is crucial to note that in 3 dimensions, Einstein 2-forms are in one-to-one correspondence with the curvature 2-forms:

$$G_a \equiv * \left(Ric_a - \frac{\mathcal{R}}{2} e_a \right) = -\frac{1}{2} \epsilon_{abc} R^{bc}. \quad (3)$$

We now consider the following Lagrangian density 3-form

$$\mathcal{L} = \mathcal{L}_{TMG} + \mathcal{L}_2 + \mathcal{L}_C, \quad (4)$$

where

$$\begin{aligned} \mathcal{L}_{TMG} &= \frac{1}{\mu} \left(\omega^a_b \wedge d\omega^b_a + \frac{2}{3} \omega^a_b \wedge \omega^b_c \wedge \omega^c_a \right) \\ &+ \frac{1}{2K} R^{ab} \wedge *e_{ab} + \Lambda *1 \end{aligned} \quad (5)$$

is the Lagrangian density of the topologically massive gravity (TMG);

$$\mathcal{L}_2 = \alpha R^{ab} \wedge *R_{ab} + \beta Ric^a \wedge *Ric_a + \gamma \mathcal{R}^2 *1 \quad (6)$$

is a generic quadratic curvature invariant with three coupling constants α , β , and γ . In 3 dimensions, one of the terms above can be made redundant by making use of the identity

$$R^{ab} \wedge *R_{ab} - 2Ric^a \wedge *Ric_a + \frac{1}{2} \mathcal{R}^2 *1 = 0. \quad (7)$$

We prefer to work without loss of generality, keeping the coupling constants α , β and γ , with

$$\mathcal{L}_2 = \left(\alpha + \frac{\beta}{2} \right) R^{ab} \wedge *R_{ab} + \left(\gamma + \frac{\beta}{4} \right) \mathcal{R}^2 *1 \quad (8)$$

for reasons of technical ease. Finally,

$$\mathcal{L}_C = T^a \wedge \lambda_a + \frac{\nu}{2} \lambda_a \wedge \lambda_b \wedge *e^{ab} \quad (9)$$

is the constraint Lagrangian density 3-form, which in case $\nu = 0$ imposes the constraint that the connection is the torsion-free Levi-Civita connection. On the other hand, if $\nu \neq 0$, the contortion 1-forms would be related with the Lagrange multiplier 1-forms in a trivial way. All previously studied models such as TMG, NMG or MMG are covered as sub-cases with the choice of the action above.

The variation of the action with respect to the Lagrange multipliers yield

$$T^a = -\nu \lambda_b \wedge *e^{ab} \iff K_{ab} = \nu \epsilon_{abc} \lambda^c. \quad (10)$$

We put this expression into the connection variation equations and solve them algebraically for the Lagrange multiplier 1-forms

$$\lambda_a = Q^{-1} \left(-\frac{2}{\mu} Y_a + (2\alpha + \beta) W_a + \left(2\gamma + \frac{\beta}{2} \right) (\iota_a * d\mathcal{R}) \right) \quad (11)$$

with

$$Q = \frac{1}{2} - \frac{\nu}{2K} - \nu \left(2\gamma + \frac{\beta}{2} \right) \mathcal{R}. \quad (12)$$

Then the final form of Einstein field equations are given as follows:

$$\begin{aligned} & - \left(\frac{1}{2K} + \left(2\gamma + \frac{\beta}{2} \right) \mathcal{R} \right) G_a + \left(\Lambda - \left(\gamma + \frac{\beta}{4} \right) \mathcal{R}^2 \right) *e_a \\ & + \left(\alpha + \frac{\beta}{2} \right) (\iota_a R^{bc} \wedge *R_{bc} - R^{bc} \wedge \iota_a *R_{bc}) \\ & + D\lambda_a + \frac{\nu}{2} \lambda^b \wedge \lambda^c *e_{abc} = 0, \end{aligned} \quad (13)$$

where we used further abbreviations

$$\begin{aligned} Y_a &= Ric_a - \frac{1}{4} e_a \mathcal{R}, \\ W_a &= \iota^b (D *R_{ba}) - \frac{1}{4} e_a (\iota^b \iota^c D *R_{cb}). \end{aligned} \quad (14)$$

We note that Einstein field equations above include among other terms, the Cotton-Schouten 2-forms

$$C_a \equiv DY_a = D \left(Ric_a - \frac{1}{4} \mathcal{R} e_a \right) \quad (15)$$

that involve third derivatives of the metric components and the 2-forms

$$D_a \equiv DW_a = D \left(\iota^b (D *R_{ba}) - \frac{1}{4} e_a (\iota^b \iota^c D *R_{cb}) \right) \quad (16)$$

that involve fourth derivatives of the metric components. It will be instructive to re-write Einstein field equations for two special cases that are of recent interest.

New Massive Gravity (NMG): $\frac{1}{\mu} \rightarrow 0$, $\Lambda = 0$, $\alpha = 0$, $\beta = 1$, $\gamma = -\frac{3}{8}$, $\nu = 0$.

$$\begin{aligned} & - \left(\frac{1}{K} - \frac{1}{2} \hat{\mathcal{R}} \right) \hat{G}_a + \frac{1}{8} \hat{\mathcal{R}}^2 *e_a + 2\hat{D}_a - \frac{1}{2} \hat{D}(\iota_a *d\hat{\mathcal{R}}) \\ & + \frac{1}{2} \left(\iota_a \hat{R}^{bc} \wedge *\hat{R}_{bc} - \hat{R}^{bc} \wedge \iota_a *\hat{R}_{bc} \right) = 0. \end{aligned} \quad (17)$$

A hat $\hat{\cdot}$ over a tensor means that it refers to the unique, torsion-free Levi-Civita connection of the metric.

Minimal Massive Gravity (MMG): $\alpha = \beta = \gamma = 0$, $\nu \neq 0$.

$$\begin{aligned} & - \frac{1}{K} G_a + \Lambda *e_a - \frac{4K}{\mu(K - \nu)} C_a \\ & + \frac{8K^2 \nu}{\mu^2 (K - \nu)^2} Y^b \wedge Y^c \wedge *e_{abc} = 0, \end{aligned} \quad (18)$$

$$K_{ab} = -\frac{4K\nu}{\mu(K - \nu)} \epsilon_{abc} Y^c. \quad (19)$$

In order to proceed any further in the study of a 3D quantized theory of gravity based on our model, its background solutions should be found. Towards that end, here we consider three-dimensional non-Riemannian space-times of constant curvature and constant torsion. We also conveniently work with coordinate independent methods. That is to say, we evaluate curvatures and their derivatives without differentiating any functions. The relevant differential geometric techniques are explained elsewhere [14]. Our starting point will be the structure equations satisfied by an orthonormal set of left-invariant basis 1-forms $\{e^a\}$ on AdS_3 :

$$\begin{aligned} e^0 &= \cosh \chi dt + \cos t \sin t \sinh \chi d\chi + \sin^2 t \sinh^2 \chi d\theta, \\ e^1 &= \sinh \chi \cos \theta dt \\ &\quad + (\cos t \sin t \cosh \chi \cos \theta + \sin^2 t \sin \theta) d\chi \\ &\quad + \sin^2 t \sinh \chi (\cosh \chi \cos \theta - \cot t \sin \theta) d\theta, \\ e^2 &= \sinh \chi \sin \theta dt \\ &\quad + (\cos t \sin t \cosh \chi \sin \theta - \sin^2 t \cos \theta) d\chi \\ &\quad + \sin^2 t \sinh \chi (\cosh \chi \sin \theta + \cot t \cos \theta) d\theta. \end{aligned} \quad (20)$$

It is now straightforward to verify i) that in the local coordinate chart $\{t, \chi, \theta\}$, the metric tensor becomes

$$\begin{aligned} g_{AdS_3} &= -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 \\ &= -dt^2 + \sin^2 t (d\chi^2 + \sinh^2 \chi d\theta^2), \end{aligned} \quad (21)$$

and ii) that these basis 1-forms satisfy the first Cartan structure equations

$$de^a = -\epsilon^a_{bc} e^b \wedge e^c. \quad (22)$$

Thus we determine the Levi-Civita connection 1-forms (with a metric that is rescaled by an overall constant factor ρ^2)

$$\hat{\omega}^a_b = -\frac{1}{\rho} \epsilon^a_{bc} e^c, \quad (23)$$

and the corresponding curvature 2-forms are

$$\hat{R}^a_b = -\frac{1}{\rho^2} e^a \wedge e_b. \quad (24)$$

Now, we set the torsion 2-forms to be

$$T^a = \frac{2}{\sigma} * e^a, \quad \sigma^2 \neq \rho^2 \iff K^a_b = -\frac{1}{\sigma} \epsilon^a_{bc} e^c. \quad (25)$$

Therefore the full curvature 2-forms turn out to be

$$R^a_b = \left(\frac{\rho^2 - \sigma^2}{\rho^2 \sigma^2} \right) e^a \wedge e_b. \quad (26)$$

Their contractions give

$$Ric_a = 2 \left(\frac{\rho^2 - \sigma^2}{\rho^2 \sigma^2} \right) e_a, \quad \mathcal{R} = 6 \left(\frac{\rho^2 - \sigma^2}{\rho^2 \sigma^2} \right). \quad (27)$$

Let us rename our variables as

$$\frac{1}{\sigma} \equiv \xi, \quad \frac{1}{\rho} \equiv \eta, \quad (28)$$

and let

$$a \equiv \frac{\mu}{4} \left(\frac{1}{\nu} - \frac{1}{K} \right). \quad (29)$$

Then our coupled system of equations reduce to the following equations of two algebraic curves in the $(\xi\eta)$ -plane:

$$(2\alpha + 4\beta + 12\gamma)\xi \frac{(\xi^2 - \eta^2)}{a^2} + \frac{(\xi - a)^2}{a^2} - \frac{\eta^2}{a^2} = 1 \quad (30)$$

and

$$-(2\alpha - 2\beta - 12\gamma) \frac{(\xi^2 - \eta^2)^2}{\Lambda} + K \left(\frac{1}{\nu} - \frac{1}{K} \right) \frac{\xi^2}{K\Lambda} + \frac{\eta^2}{K\Lambda} = 1. \quad (31)$$

We take $K > 0$ without loss of generality, since at this point our model is not yet coupled to matter. We also note that solutions come in pairs with values $\eta \leftrightarrow -\eta$. A change in sign of η means going from one orientation of the co-basis to the other, or vice versa. In what follows, we restrict attention without loss of generality to the cases $\eta > 0$. Then the desired background solutions are going to be specified by the intersection points of these two curves. As an example, in the particular case of MMG ($\alpha = \beta = \gamma = 0$), these curves reduce to conic sections and their intersections can be found. Given $K > 0$ and $\Lambda = -l^2 < 0$, there exists no solution in the range $\nu \in [0, K]$. If $\nu \notin [0, K]$, then for $\mu > 0$ solutions exist with

$$0 \leq \xi \leq \sqrt{\left| \frac{K - \nu}{\nu} \right|} K l^2 \quad (32)$$

and

$$-\sqrt{\left| \frac{K - \nu}{\nu} \right|} K l^2 \leq \xi \leq -\frac{\mu}{2K} \left| \frac{K - \nu}{\nu} \right| \leq 0, \quad (33)$$

in case $2K^2 l^2 < \mu$. On the other hand, if $\mu < 0$ then solutions exist for

$$\xi \leq -\sqrt{\left| \frac{K - \nu}{\nu} \right|} K l^2 \quad (34)$$

and

$$0 < \sqrt{\left| \frac{K - \nu}{\nu} \right|} K l^2 \leq \xi \leq \frac{|\mu|}{2K} \left| \frac{K - \nu}{\nu} \right| \quad (35)$$

in case $2K^2 l^2 < |\mu|$.

We want to conclude with some brief remarks. Firstly, the generic quadratic curvature term in the action that is usually considered in recent literature is the one that involves the square of the Ricci tensor rather than the square of the Riemann tensor. Both of these are valid choices, however, with our choice the results can be presented in a more concise and geometrically transparent manner. Secondly, the notion of 3-dimensional space-times of constant curvature and constant torsion has been overlooked for a long time. It provides a new pathway for the construction of physically relevant ground state configurations for

non-Riemannian 3D-gravity models. Thirdly, the most general background solutions in this set-up would be given by the points of intersection of a cubic and a quartic algebraic curve. It is always possible to carry out a graphical analysis to determine their intersections.

REFERENCES

- [1] DESER S., JACKIW R. and TEMPLETON S., *Phys. Rev. Lett.*, **48** (1982) 975.
- [2] DESER S., JACKIW R. and TEMPLETON S., *Ann. Phys.*, **140** (1982) 372; **185** (1988) 406.
- [3] BANADOS M., TEITELBOIM C. and ZANELLI J., *Phys. Rev. Lett.*, **69** (1992) 1849.
- [4] BANADOS M., HENNEAUX M., TEITELBOIM C. and ZANELLI J., *Phys. Rev. D*, **48** (1993) 1506.
- [5] BERGSHOEFF E. A., HOHM O., and TOWNSEND P. K., *Phys. Rev. Lett.*, **102** (2009) 201301.
- [6] DESER S., *Phys. Rev. Lett.*, **103** (2009) 101302.
- [7] BERGSHOEFF E. A., HOHM O., MERBIS W., ROUTH A. J. and TOWNSEND P. K., *Class. Quantum Grav.*, **31** (2014) 145008.
- [8] DERELI T. and TUCKER R. W., *Class. Quantum Grav.*, **5** (1988) 951.
- [9] CARTAN É., *C. R. Acad. Sci. (Paris)*, **174** (1922) 593.
- [10] TRAUTMAN A., *Einstein-Cartan Theory*, *Encyclopedia of Mathematical Physics*, edited by FRANÇOISE J.-P., NABER G. L. and TSOU S. T., Vol. **2** (Elsevier) 2006, p. 189.
- [11] LAZAR M. and HEHL F. W., *Found. Phys.*, **490** (2010) 1298.
- [12] DERELI T. and VERÇİN A., *Philos. Mag. B*, **56** (1987) 625.
- [13] DERELI T. and VERÇİN A., *Philos. Mag. B*, **64** (1991) 509.
- [14] DERELI T. and YETİŞMİŞOĞLU C., *New improved massive gravity and three dimensional spacetimes of constant curvature and constant torsion*, preprint, arXiv:1604.00463 [gr-qc].