Subcontracting with Availability Guarantees: Production Control and Capacity Decisions

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February 20, 2004

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Abstract

We present a simplified model of a system with a producer, a subcontractor, and a random demand. The demand level alternates between a high level and a low level with exponential switching times. The producer does not have enough capacity to meet the high demand. Therefore, it either produces to stock in advance or uses a subcontractor to receive additional capacity when it needs. The subcontractor serves a number of manufacturers and guarantees a *long term availability* that is defined as the long-term probability that the subcontractor will be available when it is requested, to each manufacturer. Therefore, a manufacturer may not receive the requested capacity from the subcontractor immediately and waits until the subcontractor becomes available. The times that the subcontractor is available and not available are also exponential random variables. The producer uses a threshold-type policy that depends on the state of the inventory/backlog to decide how much to produce and how much to request from the subcontractor. This system is modelled analytically based on a stochastic flow rate control problem with continuous flow and discrete states in a Markovian setting. A numerical analysis of the model is used to analyze the effects of guaranteed availability on the manufacturer's and subcontractor's performances. Extensions to the producer's and subcontractor's capacity decisions and the subcontractor's pricing decisions are also discussed.

1. Introduction

Subcontracting and outsourcing have been the subject of a growing number of studies in recent years. In this study, we focus on using subcontracting to achieve capacity flexibility. This is an effective strategy to cope with demand variability. The main motivation in this study comes from the textile, apparel, retail channel. In recent years, retailers adopt lean retailing practices to place a larger fraction of their orders during the season (Abernathy, Dunlop, Hammond, and Weil 1999). This change shifts the risks associated with carrying too much or too little inventory from retailers to manufacturers. In order to respond quickly to the retailer demand, a manufacturer can produce well in advance to stock or increase its capacity to reduce the lead time. Alternatively, using subcontractors can be an attractive option for manufacturers to respond quickly to these retailers without carrying too much or too little inventory. Higher prices associated with subcontractors that are located near the market can be justified by reduced inventory carrying, lost sales, and markdown costs (Abernathy, Dunlop, Hammond, and Weil 2000).

Similar to the shift of the risks from a retailer to a manufacturer, this strategy may also introduce risks for subcontractors. More specifically, reserving the capacity for each customer causes a lower utilization of a subcontractor's capacity. This would increase the costs of subcontracting and introduce inefficiencies in the supply chain. Alternative ways of managing manufacturer-subcontractor relations through availability guarantees may eliminate this problem.

The objective of this study is to analyze an agreement that guarantees a certain level of long-term availability of the subcontractor's capacity to a producer. Accordingly, the subcontractor does not reserve a capacity for each of its customers and therefore it may not be available at a time when a manufacturer requests production. However the subcontractor promises that in the long run, the probability that a manufacturer will find the subcontractor available will be at an agreed level. Moreover, the subcontractor informs the manufacturer the average response time to start production in case it is not available.

In this agreement, while the producer benefits from the reduced inventory and backlog costs, the subcontractor benefits from pooling the demand, i.e., sharing the risks, and therefore serving more producers with a given level of capacity.

Although the primary motivation of this study is applications in production, similar agreements exist in call center outsourcing as partial-outsourcing contracts with service level agreements. Furthermore, the maximum delivery time agreements with logistics service providers can also be thought as using subcontractors that are not always available. In both of these cases, the subcontractors, e.g. the call center and logistics service providers use their capacity more efficiently by benefiting from pooling.

We use a simplified model to analyze the complex relationship between a producer and supplier and investigate how the relationship impacts a number of producer and supplier decisions. The primary producer decisions are how much to produce at a given time, how much to subcontract at a given time depending on inventory and backlog levels. A secondary producer decision is how much capacity to invest in at the start of the planning horizon.

The primary decision for the supplier is determining the capacity required to promise an agreed level of long-term availability to each producer with the average length of time it takes for the subcontractor to become available to the manufacturer if it is unavailable at the time of request. A secondary decision is the unit price charged by the subcontractor to the producer. The subcontractor meets the demand of many identical producers so that the benefits obtained from the pooling of demand uncertainty are captured in the subcontractor's decisions. Two sources of uncertainty exist in the model. First, demand randomly switches between a fixed high level and a fixed low level. Second, the subcontractor may or may not be available to the producer when needed (i.e., the producer may have to wait for the subcontractor availability). Both the producer and subcontractor are profit-maximizing firms.

These decisions are analyzed by utilizing an analytic model that is based on a stochastic flow-rate control problem in a continuous flow and discrete state space Markovian system. The optimal production and subcontracting policy of the producer is characterized and the performance of the system governed by this optimal policy is analyzed numerically.

2. Past Work

2.1 Subcontracting and Outsourcing

Subcontracting and capacity investment decisions are studied in a number of recent studies. A Brownian motion approximation for the optimal subcontracting policy for an $M/M/1$ system is used to study capacity, inventory, and subcontracting jointly in (Bradley and Glynn

2002). As an extension, Bradley (2002b) presents the dynamic control of an $M/M/1$ queueing system where two supply sources are available. Under the assumption that a basestock level is used for production, it is shown that a threshold policy is optimal where the second source is activated once a given level is reached. Then, a Markov chain analysis is used to determine the optimal base-stock levels. The key differences between this work and Bradley's are that the current work assumes a subcontractor that is not always available; whereas, Bradley has a Markovian subcontracting source that is always available (but takes an exponential time to produce any one item). Further, the current work provides an extension to the subcontractor's capacity and pricing decisions that are not considered by Bradley.

Kouvelis and Milner (2002) present a two-stage model where the effects of demand and supply uncertainties on capacity expansion decisions are studied. de Kok (2000) presents a model where a company decides on its capacity reservation according to a strategy based on a periodic review order-up-policy. In this study, excess capacity needs are outsourced.

The competition issues related to subcontracting, capacity, and investment decisions are discussed in a two-period competitive stochastic investment game by Van Mieghem (1999). Cachon and Harker (2002) also present a game-theoretic model for the effects of outsourcing in competition. Atamturk and Hochbaum (2001) present a deterministic lot sizing model to determine the capacity and subcontracting capacity to satisfy non-stationary demand over a finite horizon.

Gutierrez and Paul (2000) present a model that captures the effects of risk pooling and subcontracting in a project management setting. Gaimon (1994) presents a model that investigates subcontracting as an alternative to capacity expansion. She also examines the effects of using subcontracting on pricing of services. Bertrand and Sridharan (2001) study various rules of subcontracting in a make-to-order manufacturing system.

2.2 Stochastic Flow Rate Control

The methodology of this study is closely related to a line of work in stochastic flow-rate control of manufacturing systems. An optimal flow-rate control problem for a failure prone machine subject to a constant demand source was introduced by Olsder and Suri (1980) and Kimemia and Gershwin (1983). The single-part-type, single-machine problem was analyzed in detail by Bielecki and Kumar (1988). The optimal control is a hedging point policy where the machine operates at its maximum rate until the inventory reaches a certain level; and then it operates at a rate that keeps the inventory at this level. Hedging point control policies are optimal or near optimal for a range of manufacturing system models. Most of these studies assume a constant demand source. Only a few consider optimal production control problems with random demand, including Fleming, Sethi, and Soner (1987), Krichagina, Lou, and Taksar (1994), Perkins and Srikant (2001), and Tan (2002b). For an overview of the dynamic programming formulations of factory scheduling and inventory control and a comprehensive list of references, see Gershwin (1994).

A number of studies extended the flow-rate control problem to investigate subcontracting for single machine-single product cases, e.g., (Hu 1995), (Huang, Hu, and Vakili 1996), (Tan 2002a), (Tan and Gershwin 2004).

All the previous studies consider subcontracting from the producers' point of view and therefore assume always available subcontractors. The main contribution of this study is to

present a model where the subcontractor may not be available due to serving other customers. This extension allows us to examine the capacity decisions of subcontractors to guarantee a long-term availability. We characterize the manufacturer's optimal strategy of production and subcontracting in this setting. The performance of the system when it is governed by this optimal strategy is the basis of the analysis.

The model, its assumptions, and the production control problem are introduced in Section 3.1 and 3.2. The characterization of the optimal policy is discussed in Section 3.3 and a simplified threshold-type production and subcontracting policy is proposed in 3.4. The model is analyzed in Section 4 and the average optimal profit rate is derived from the steady-state distribution of the system state in Section 5. The effects of the guaranteed availability, and the response rate are examined numerically in Section 6. The subcontractor's capacity requirements to guarantee a long term availability are investigated in 7. A number of extensions including the capacity investment decision of the manufacturer and the investment and pricing decisions of the subcontractor are discussed in 8. Finally, the conclusions are given in 9.

3. Model of Subcontracting with Guaranteed Availability

In this section, we present the assumptions of the model and introduce a subcontractor that provides additional capacity under a pre-specified availability agreement. The manufacturing plant and the subcontractor have a capacity — a maximum rate at which it can deliver. The manufacturer also has a price that it will sell the goods at. The manufacturer can also make a profit by buying the item from a subcontractor and reselling each item. We assume that the profit from reselling is less than that from the manufacturer's own production facilities.

3.1 Model and Assumptions

We consider a make-to-stock system consisting of a single manufacturing facility and a subcontractor that produces to meet the demand for a single item. Production, demand, inventory, and backlog are all represented by continuous (real) variables. The demand rate at time t is denoted by $d(t)$. The state of the demand at time t is $D(t)$ which is either high (H) or low (L). When the demand is high, the demand rate is $d(t) = \mu_H$ and when the demand is low, the demand rate is $d(t) = \mu_L$. At time t , the amount of finished goods inventory or backlog is $x(t)$.

The times to switch from a high demand state to a low demand state and from a low demand state to a high demand state are assumed to be exponentially distributed random variables with rates λ_{HL} and λ_{LH} .

This alternating demand model has been used to capture demand variability in continuous flow models. The distribution of the accumulated demand generated by the switching demand source is asymptotically normal with the average demand rate and the limiting variance rate given as

$$Ed = \mu_H e + \mu_L(1 - e), \quad Vd = \frac{2(\mu_H - \mu_L)^2(1 - e)e^2}{\lambda_{LH}} \quad (1)$$

where $e = \frac{\lambda_{LH}}{\lambda_{HL} + \lambda_{LH}}$ is the percentage of the time the demand is high (Tan 1997). In this study, the asymptotic coefficient of variation of the demand, defined as $cv = \sqrt{Vd}/Ed$, is used as a summary measure of the demand variability.

At time t , the producer produces finished goods at rate $u_0(t)$. The maximum production rate of the producer is $\underline{u}_0$, $0 \leq u_0(t) \leq \underline{u}_0$. We assume that the producer has sufficient capacity to meet the lower demand but cannot meet the high demand, i.e., $\mu_L < \underline{u}_0(t) < \mu_H$. Note that if $\underline{u}_0(t) > \mu_H$ then the problem is trivial, the manufacturer does not use a subcontractor and meets the demand with no inventory or backlog accumulation. Therefore, it either produces to stock in advance or uses a subcontractor to receive additional capacity when it needs to meet demand fluctuations.

At time t , the producer also requests the subcontractor to supply materials at a rate of $u_1(t)$. The maximum rate at which the subcontract supplies the materials is $\underline{u}_1$, $0 \leq u_1(t) \leq \underline{u}_1$. We assume that it is possible to meet the high demand when the subcontractor is also used, i.e., $\underline{u}_0 + \underline{u}_1 > \mu_H$. Furthermore, the in-house production and the subcontracted production is sufficient to meet the demand in the long run.

When the producer asks for additional capacity from the subcontractor, the subcontractor may not be available because it may be providing capacity to other customers. The state of the subcontractor at time t is denoted with $S(t)$ which is O if it is available (on) and F if it is not available (off) at time t . We assume that the periods where the subcontractor is available and not available follow an alternating renewal process with exponential switching times with rates p and r respectively.

The subcontractor guarantees a long-term availability level of β . β is the long-term probability that at a given instant the subcontractor will be available to start production for the manufacturer. Following the ergodicity of the process, β is also the fraction of the time that the subcontractor is available, i.e., $\beta = \frac{r}{p+r}$. The subcontractor also informs the manufacturer about the average time it takes until it starts providing additional capacity when it is not available, $1/r$.

The profit coefficient (dollars per unit) for the goods produced in the factory is J_0 and the profit coefficient (dollars per unit) when the subcontractor is used is J_1 where $J_0 > J_1 > 0$. The inventory carrying cost is g^+ (dollars per unit per time) and the corresponding backlog cost is g^- . We do not consider any fixed costs associated with starting or stopping using a subcontractor.

3.2 Production Control Problem of the Producer

The decision variables are the rate at which the goods are produced at the plant at time t , $u_0(t)$, and the rate at which the subcontractor is requested to supply goods at time t , $u_1(t)$. The profit function to be maximized is the difference between the average profit generated through production and subcontracting and the average inventory carrying and backlogging costs. A linear inventory carrying cost function is assumed, i.e.,

$$g(x) = \begin{cases} g^+x & \text{if } x \geq 0 \\ -g^-x & \text{if } x < 0 \end{cases}$$

The production control problem is

$$V = \max_{u_0(t), u_1(t)} \Pi_0 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T E [J_0 u_0(t) + J_1 u_1(t) - g(x(t))] dt \quad (2)$$

subject to

$$\frac{dx(t)}{dt} = u_0(t) + u_1(t) - d(t) \quad (3)$$

$$0 \leq u_0(t) \leq \underline{u}_0. \quad (4)$$

$$0 \leq u_1(t) \leq \underline{u}_1 s(t). \quad (5)$$

$$d(t) = \begin{cases} \mu_H & \text{if } D(t) = H \\ \mu_L & \text{if } D(t) = L \end{cases} \quad (6)$$

$$s(t) = \begin{cases} 1 & \text{if } S(t) = O \\ 0 & \text{if } S(t) = F \end{cases} \quad (7)$$

$$\text{Markov dynamics for } D(t) \text{ with rates } \lambda_{HL} \text{ from H to L and } \lambda_{LH} \text{ from L to H} \quad (8)$$

$$\text{Markov dynamics for } S(t) \text{ with rates } p \text{ from O to F and } r \text{ from F to O} \quad (9)$$

3.3 The Structure of the Optimal Production and Subcontracting Policy

Available results in the literature for similar problems including (Hu 1995), (Huang, Hu, and Vakili 1996), (Tan 2002a) suggest a hedging-type policy for the production control problem given above. Since the state transitions follow a continuous-time Markov chain with four states, it is expected that the optimal policy will have a total of four thresholds, one for each state. Considering the feasibility of these states, only three of them will appear as hedging levels or thresholds. The full characterization of the optimal policy is not given here but can be developed by following the methodology of the above-mentioned studies.

There are three critical levels; two hedging points and one threshold in the optimal policy of deciding production and subcontracting. The upper level determines how much finished goods inventory to be accumulated and the lower thresholds determine when to start and when to stop subcontracting. These levels are determined optimally to maximize the profit. Therefore, the solution of Problem (2)-(9) is totally characterized by these levels.

3.4 A Simplified Production and Subcontracting Policy

The optimal policy stated in the previous section requires full information about the state of the demand. We use an alternating demand model between a high state and a low state only to capture the demand variability in our model. In other words, when we want to model a demand distribution, we can match its two moments by using equation (1) and determining μ_H and μ_L . Therefore, having thresholds depending on these constant levels is not implementable in practice.

In light of this discussion on the observability of the system, we propose using only two levels; one for deciding on the safety stock and another one to decide on when to call the subcontractor. Therefore, a manufacturer can control its system by observing the inventory and backlog level. Moreover, it is observed in the literature that using a fewer number of hedging levels does not deteriorate the performance considerably in similar models (Tan 2002b).

Accordingly, let Z_0 and Z_1 denote the upper and lower thresholds. The manufacturer produces with the maximum production rate until the finished goods inventory reaches Z_0 . At this level, the manufacturing facility produces at the demand rate to keep x at this level. That is,

$$u_0(t) = \begin{cases} \mu_L & x(t) = Z_0 \\ \underline{u}_0 & x(t) < Z_0 \end{cases} \quad (10)$$

The subcontractor is used only when the surplus/backlog level is less than or equal to Z_1 . When x reaches Z_1 , if the subcontractor is available, it is requested to supply goods at a rate of $\mu_H - \underline{u}_0$ to keep x at this level. If the subcontractor is not available when x reaches Z_1 , x keeps decreasing. When the subcontractor becomes available, the manufacturer requests from the subcontractor to supply goods at the subcontractor's maximum capacity $\underline{u}_1$:

$$u_1(t) = \begin{cases} 0 & x(t) > Z_1 \text{ or } S(t) = F \\ \mu_H - \underline{u}_0 & x(t) = Z_1 \text{ and } S(t) = O \\ \underline{u}_1 & x(t) < Z_1 \text{ and } S(t) = O \end{cases} \quad (11)$$

Figure 1 depicts the sample path of a system where the above production and subcontracting policy is used.

4. Analysis of the Model

In this section, we calculate the steady-state probability distribution of x , D , and S assuming that the system is operated under the policy of Section 3.4. In Section 5, we evaluate the expected profit (as well as other performance measures). Then we find the values of Z_0 and Z_1 that maximize the expected profit.

4.1 Probability distribution

When the surplus/backlog x is not equal to the upper or lower levels (Z_0 or Z_1), the system is said to be in the *interior* region. The interior region $Z_1 < x < Z_0$ is referred as Region 1 and the interior region $x < Z_1$ is referred as Region 2.

Figure 1: Sample path for a system with a subcontractor ($\mu_H = 1.5$, $\mu_L = 0$, $\lambda_{LH} = 0.2$, $\lambda_{HL} = 0.13$, $\underline{u}_0 = 1$, $\underline{u}_1 = 3$, $Z_0 = 2$, $Z_1 = -1$, $p = 0.07$, $r = 0.07$)

Since the subcontractor state does not affect the decisions in Region 1, the system state at time t is $W_1(t) = (x(t), D(t))$ where $Z_1 < x(t) < Z_0$, $D(t) \in \{H, L\}$. The time-dependent system state probability distribution for Region 1, $\mathcal{F}_1(t, x, D)$, is defined as

$$\mathcal{F}_1(t, x, D) = \mathbf{prob}[x(t) \leq x, D(t) = D]; \quad t \geq 0. \quad (12)$$

In Region 2, the dynamics of the system depends on the state of the subcontractor. The state of the state at time t in Region 2 is $W_2(t) = (x(t), D(t), S(t))$ where $x(t) < Z_1$, $D(t) \in \{H, L\}$, and $S(t) \in \{O, F\}$.

$$\mathcal{F}_2(t, x, D, S) = \mathbf{prob}[x(t) \leq x, D(t) = D, S(t) = S]; \quad t \geq 0. \quad (13)$$

The time-dependent system state density functions are defined as

$$f_1(t, x, D) = \frac{\partial \mathcal{F}_1(t, x, D)}{\partial x} \quad t \geq 0, \quad Z_1 < x(t) < Z_0. \quad (14)$$

and

$$f_2(t, x, D, S) = \frac{\partial \mathcal{F}_2(t, x, D, S)}{\partial x} \quad t \geq 0, \quad x < Z_1. \quad (15)$$

We assume that the process is ergodic and, thus, the steady-state density functions exist. It is possible to show ergodicity by first considering a discrete approximation of the system, i.e., where the buffer may assume only certain discrete levels. This leads to a Markov chain model where all of the states constitute a single communicating class. Furthermore, it becomes evident from the self-loops that the chain is aperiodic. The present continuous model can be shown to be the limiting case of the discrete model where the number of states of the discrete Markov chain model approaches infinity. However, a rigorous proof of ergodicity will not be given in this paper.

The steady-state density functions are defined as

$$f_1(x, D) = \lim_{t \rightarrow \infty} f_1(t, x, D), \quad Z_1 < x(t) < Z_0. \quad (16)$$

$$f_2(x, D, S) = \lim_{t \rightarrow \infty} f_2(t, x, D, S), \quad x(t) < Z_1. \quad (17)$$

The steady-state probabilities that the finished goods inventory is equal to the hedging level Z_0 and the lower level Z_1 are denoted by P_0 and P_1 .

4.2 Dynamics of Region 1

In Region 1, the steady-state differential equations for $f_1(x, H)$ and $f_1(x, L)$ can be written as

$$(\underline{u}_0 - \mu_H) \frac{df_1(x, H)}{dx} = -\lambda_{HL} f_1(x, H) + \lambda_{LH} f_1(x, L) \quad (18)$$

$$(\underline{u}_0 - \mu_L) \frac{df_1(x, L)}{dx} = \lambda_{HL} f_1(x, H) - \lambda_{LH} f_1(x, L) \quad (19)$$

where $Z_1 < x < Z_0$.

4.3 Dynamics of Region 2

In Region 2, the steady-state differential equations for $f_2(x, H, O)$, $f_2(x, H, F)$, $f_2(x, L, O)$, and $f_2(x, L, F)$ can be written as

$$(\underline{u}_0 + \underline{u}_1 - \mu_H) \frac{df_2(x, H, O)}{dx} = -(\lambda_{HL} + p)f_2(x, H, O) + \lambda_{LH}f_2(x, L, O) + rf_2(x, H, F) \quad (20)$$

$$(\underline{u}_0 + \underline{u}_1 - \mu_L) \frac{df_2(x, L, O)}{dx} = \lambda_{HL}f_2(x, H, O) - (\lambda_{LH} + p)f_2(x, L, O) + rf_2(x, L, F) \quad (21)$$

$$(\underline{u}_0 - \mu_H) \frac{df_2(x, H, F)}{dx} = pf_2(x, H, O) - (\lambda_{HL} + r)f_2(x, H, F) + \lambda_{LH}f_2(x, L, F) \quad (22)$$

$$(\underline{u}_0 - \mu_L) \frac{df_2(x, L, F)}{dx} = pf_2(x, L, O) + \lambda_{HL}f_2(x, H, F) - (\lambda_{LH} + r)f_2(x, L, F) \quad (23)$$

where $x < Z_1$.

4.4 Probabilities at the Hedging Levels

The steady-state probability that the surplus is equal to Z_0 is also the time average of the duration at Z_0 . Since Z_0 can be reached only when the demand is low, the total duration at Z_0 is equal to the number of times the process reaches Z_0 times the average duration at each visit which is $\frac{1}{\lambda_{LH}}$. Then

$$P_0 = (\underline{u}_0 - \mu_L)f_1(Z_0, L) \frac{1}{\lambda_{LH}} \quad (24)$$

In order to derive P_1 , note that, the process stays at the lower level Z_1 only when the demand is at a high level and the subcontractor is available. For all the other cases, x crosses Z_1 and enters the other region without staying at this level. Starting in Region 1, if the demand is high, x decreases and reaches Z_1 . At this time, the subcontractor is requested to supply additional capacity. However, since the subcontractor can be unavailable at the time of the request, only a fraction of level crossings from Region 1 results in a duration at Z_1 . The steady-state probability that the subcontractor will be available when x reaches Z_1 is the guaranteed long-run availability of the subcontractor β .

However, when the demand is high and the subcontractor is available, x starts increasing in Region 2 until it reaches Z_1 . At this time, the subcontractor is requested to supply at a level that keeps x at Z_1 .

The process stays at Z_1 until either the demand switches to a low level and thus x increases and enters into Region 1, or the subcontractor becomes unavailable and x decreases and enters into Region 2. Therefore the average time spent at Z_1 during each visit is $\frac{1}{\lambda_{HL} + p}$.

Therefore, the steady-state probability that $x = Z_1$ is

$$P_1 = (\beta(\mu_H - \underline{u}_0)f_1(Z_1, H) + (\underline{u}_0 + \underline{u}_1 - \mu_H)f_2(Z_1, H, O)) \frac{1}{\lambda_{HL} + p} \quad (25)$$

5. Solution of the Model

In order to solve the differential equations given in (18)-(23), six boundary conditions are required. A level crossing analysis is employed to derive some of these boundary equations.

5.1 Boundary Conditions

First, note that at any given level of the finished goods inventory, the number of upward crossings must be equal to the number of downward crossings. In Region 1, this equality yields the first boundary condition:

$$(\mu_H - \underline{u}_0)f_1(x, H) = (\underline{u}_0 - \mu_L)f_1(x, L) \quad (26)$$

Similarly, in Region 2, we have

$$(\underline{u}_0 + \underline{u}_1 - \mu_H)f_2(x, H, O) + (\underline{u}_0 + \underline{u}_1 - \mu_L)f_2(x, L, O) + (\underline{u}_0 - \mu_L)f_2(x, L, F) = (\mu_H - \underline{u}_0)f_2(x, H, F) \quad (27)$$

The level crossings at Z_1 yield the third boundary condition. Let us consider the number of level crossings into Region 1 from Z_1 when the demand is low. This crossing is possible either coming directly from Region 2 or from $x = Z_1$. If the demand is low, x starts increasing in Region 2, crosses Z_1 and enters into Region 1. If x is at Z_1 , it enters into Region 1, only when the demand switches to low. Therefore the level crossing equality at Z_1 when the demand is low and x enters into Region 1 can be written as

$$\begin{aligned} (\underline{u}_0 - \mu_L)f_1(Z_1, L) &= (\underline{u}_0 + \underline{u}_1 - \mu_L)f_2(Z_1, L, O) + (\underline{u}_0 - \mu_L)f_2(Z_1, L, F) \\ &\quad + (\beta(\mu_H - \underline{u}_0)f_1(Z_1, H) + (\underline{u}_0 + \underline{u}_1 - \mu_H)f_2(Z_1, H, O)) \frac{\lambda_{HL}}{\lambda_{HL} + p} \end{aligned} \quad (28)$$

The next boundary condition is the normalization condition. That is, the sum of the probabilities that the process is in Region 1 and in Region 2 with the probabilities P_0 and P_1 is equal to one:

$$P_0 + P_1 + \int_{Z_1}^{Z_0} \sum_D f_1(x, D) dx + \int_{-\infty}^{Z_1} \sum_{D,S} f_2(x, D, S) dx = 1 \quad (29)$$

where $D \in \{H, L\}$, and $S \in \{O, F\}$.

The remaining boundary equations are determined by setting the weights of the negative eigenvalues in the solution of the dynamics in Region 2 following the feasibility assumption.

These boundary equations yield the solution of the steady-state probability density functions. Since P_0 and P_1 are also determined by these density functions following (24) and (25), the complete steady-state probability distribution of the process $(x(t), D(t), S(t))$ is obtained. In the next section, this distribution is used to evaluate the objective function. Then the optimal values of Z_0 and Z_1 are determined by maximizing the objective function.

5.2 Evaluation of the Objective Function

The first term in the profit function given in (2) is the revenue generated through in-house production. Since the manufacturing plant produces with the maximum rate when $x < Z_0$ and produces with the demand rate when $x = Z_0$, the average in-house production rate Ψ_0 is given as

$$\Psi_0 = (1 - P_0)\underline{u}_0 + P_0\mu_L \quad (30)$$

Accordingly, the profit generated through in-house production is $J_0\Psi_0$.

The second term in (2) is the money generated through subcontracted production. When the subcontractor is available, it supplies goods at a rate of $\mu_H - \underline{u}_0$ at $x = Z_1$ and otherwise supplies with the maximum rate $\underline{u}_1$. When it is not available, no additional production is obtained and no money is generated. Then the average subcontracted production rate Ψ_1 is given as

$$\Psi_1 = (\mu_H - \underline{u}_0)P_1 + \int_{-\infty}^{Z_1} \underline{u}_1 (f_2(x, H, O) + f_2(x, L, O)) dx \quad (31)$$

and the profit generated through subcontracted production is $J_1\Psi_1$.

The next term in equation (2) reflects the inventory carrying costs. The average inventory level E_{WIP} is given as

$$E_{WIP} = \begin{cases} Z_0P_0 + Z_1P_1 + \int_{Z_1}^{Z_0} \sum_D x f_1(x, D) dx + \int_0^{Z_1} \sum_{D,S} x f_2(x, D, S) dx & 0 \leq Z_1 < Z_0 \\ Z_0P_0 + \int_0^{Z_0} \sum_D x f_1(x, D) dx & Z_1 < 0 \leq Z_0 \end{cases} \quad (32)$$

where $D \in \{H, L\}$, and $S \in \{O, F\}$.

Similarly, the average backlog level E_{BG} is given as

$$E_{BG} = \begin{cases} - \int_{-\infty}^0 \sum_{D,S} x f_2(x, D, S) dx & 0 \leq Z_1 < Z_0 \\ -Z_1P_1 - \int_{Z_1}^0 \sum_D x f_1(x, D) dx - \int_{-\infty}^{Z_1} \sum_{D,S} x f_2(x, D, S) dx & Z_1 < 0 \leq Z_0 \end{cases} \quad (33)$$

where $D \in \{H, L\}$, and $S \in \{O, F\}$.

Finally, the average profit rate is

$$\Pi_0 = J_0\Psi_0 + J_1\Psi_1 - g^+ E_{WIP} - g^- E_{BG}. \quad (34)$$

Once the objective function is available, the optimal values of Z_0 and Z_1 are determined by maximizing Π_0 .

6. Effects of Subcontractor's Availability and Response Rate on the Manufacturer

The analytical model presented in the preceding section can be analyzed and the results can even be given in closed-form. However, the final expressions are so lengthy that they are of limited use. Alternatively, we analyze the model numerically. Although only a single case is reported, numerous numerical experiments showed that the qualitative behavior of the system is the same as the reported results for a wide range of system parameters.

6.1 Effect of Subcontractor's Guaranteed Availability

The effect of the subcontractor's pre-specified long-term availability, β , on the performance of the system is depicted in Figure 2.

Figure 2 depicts the effect of the subcontractor's long-term availability on the profit, and in-house and subcontracted production rates of the manufacturer. The manufacturer benefits from increased β . As the guaranteed availability of the subcontractor increases, the manufacturer shifts some of its in-house production to the subcontractor and increases its profit. However, the marginal benefit of an increase in β diminishes and the profit rate of the manufacturer is determined by other factors including the capacity of the manufacturer, demand variability, production and subcontracting costs etc. In other words, the producer's profit is not very sensitive to β above a certain level. Since the information on β and r are provided to the producer, the producer can decide on the hedging levels and realize almost the same profit. As long as a right long term availability is agreed, the manufacturer is not affected adversely because of unavailability of the subcontractor.

6.2 Effect of Subcontractor's Average Response Rate

The subcontractor informs the manufacturer not only its long-term availability but also the average time it takes to start production in case it is not available upon the request of the manufacturer. The effect of the subcontractor's average response rate r on the performance of the system is depicted in Figure 3.

Figure 3 depicts the effect of the subcontractor's long-term availability on the profit, and in-house and subcontracted production rates. As the subcontractor responds faster, the producer increases its profit, and uses the subcontractor more. However, the producer is not very sensitive to the response rate above a certain level as long as r is given. Similar to the previous case, the manufacturer can adjust its thresholds to make up for longer response times to achieve the same profit. Only if the response rate is too slow, then it becomes critical and affects the profit rate negatively.

The above discussion shows that when the average profit is taken as the main performance indicator, unavailability of the subcontractor can be accommodated by the manufacturer by adjusting its thresholds, e.g. by holding more finished goods inventory or requesting the subcontractor earlier. Therefore, the manufacturer does not require capacity reservation from the subcontractor in order to achieve a given level of profit as long as they agree on a certain level of long term availability and it is informed about the average response time.

Figure 2: Effect of the Subcontractor's Long-Term Availability on the Profit, and In-house and Subcontracted Production Rates ($\mu_H = 1.5$, $\mu_L = 0$, $Ed = 1$, $cv = 2$, $r = 0.1$, $\underline{u}_0 = 0.9$, $\underline{u}_1 = 3$, $J_0 = 3$, $J_1 = 1$, $g^+ = 0.04$, $g^- = 0.04$)

Figure 3: Effect of the Subcontractor's Average Response Rate on the Profit, and In-house and Subcontracted Production Rates ($\mu_H = 1.5$, $\mu_L = 0$, $Ed = 1$, $cv = 2$, $\underline{u}_0 = 0.9$, $\underline{u}_1 = 3$, $\beta = 0.85$, $J_0 = 3$, $J_1 = 1$, $g^+ = 0.04$, $g^- = 0.04$)

7. Subcontractor's Capacity Requirements to Guarantee a Long-Term Availability

In this section, we examine the guaranteed availability on the subcontractor's investment and pricing decisions. Since the subcontractor does not reserve capacity for each producer, it can benefit from pooling the demand and serve more customers with a given level of capacity.

In order to quantify the effects of pooling, we first derive the distribution of the total demand generated by N identical producers on the subcontractor. Then based on this distribution, we determine the subcontractor's capacity required to guarantee β to each producer. Finally, we examine the capacity investment and pricing decisions. Note that this procedure can also be used to analyze N non-identical producers. In order to discuss the implications concisely, identical manufacturers are used.

7.1 Distribution of the Subcontractor's Total Demand

As Equation (11) shows, at a given time, the subcontractor provides additional capacity to a manufacturer at a rate of $\underline{u}_1$ or at a rate of $\mu_H - \underline{u}_0$. At other times, either the manufacturer does not request or the subcontractor is unavailable to supply additional capacity.

Let Γ be the instantaneous demand rate, i.e. capacity, requested by a producer. Following the processes $x(t)$, $D(t)$, and $S(t)$, Γ is random variable that takes values $\mu_H - \underline{u}_0$, $\underline{u}_1$, and 0.

The subcontractor delivers at a rate of $\mu_H - \underline{u}_0$ when x is at the lower threshold. Therefore, the steady-state instantaneous probability that the subcontractor delivers at this rate is equal to P_1 . That is,

$$\gamma_1 = \mathbf{prob}[\Gamma = \mu_H - \underline{u}_0] = P_1. \quad (35)$$

The steady-state probability that the subcontractor delivers at $\underline{u}_1$ is equal to the probability that x is in Region 2 and the subcontractor is available. Equivalently, since the average subcontracted production rate and γ_1 are known,

$$\gamma_2 = \mathbf{prob}[\Gamma = \underline{u}_1] = \frac{\Psi_1 - (\mu_H - \underline{u}_0)\gamma_1}{\underline{u}_1}. \quad (36)$$

Finally,

$$\gamma_3 = \mathbf{prob}[\Gamma = 0] = 1 - \gamma_1 - \gamma_2. \quad (37)$$

If a subcontractor serves N independent and identical manufacturers, then the total demand placed on the subcontractor will be $\underline{\Gamma} = \sum_{i=1}^N \Gamma_{(i)}$. The distribution of $\underline{\Gamma}$ will be the N -fold convolution of Γ . If the manufacturers are not identical then the convolution of the sum of N different random variables yields the distribution of $\underline{\Gamma}$.

Figure 4 shows the distribution of $\underline{\Gamma}$ for a specific case with 25, 50, 75, and 100 identical manufacturers. In this example, the subcontractor agrees on providing an additional capacity of maximum $\underline{u}_1$ at a certain cost (that determines J_1), with a long-term availability β

Figure 4: Distribution of the total subcontractor's demand from N manufacturers ($\mu_H = 1.5$, $\mu_L = 0$, $Ed = 0.75$, $cv = 2$, $\underline{u}_0 = 1$, $\underline{u}_1 = 3$, $p = 0.1$, $r = 0.9$, $J_0 = 5$, $J_1 = 2.2$, $g^+ = 0.08$, $g^- = 0.64$)

and average response rate r . Based on these values, its own capacity, and the demand variability, each manufacturer determines the thresholds optimally to maximize its profit. These thresholds then determine the distribution of Γ . In this example, $\gamma_1 = 0.1454$, $\gamma_2 = 0.0027$, and $\gamma_3 = 0.8518$.

7.2 Required Subcontractor's Capacity to Guarantee β

Once the distribution of $\underline{\Gamma}$ is available, the subcontractor can set its capability of responding to the manufacturer's demand according to the specified long-term availability level β . Since β is defined as the instantaneous probability that the subcontractor will be available upon a request of a manufacturer, a subcontractor can set its own capacity to a level, say $C(N)$ such that the probability that the accumulated demand at a given instant is greater than this capacity is less than or equal to the guaranteed availability. That is,

$$C(N) = \min\{c \mid \mathbf{prob}[\underline{\Gamma} \geq c] \leq \beta\} \quad (38)$$

Figure 5 shows the required capacity level for a subcontractor with different number of

Figure 5: Required capacity to provide a desired availability level. ($\mu_H = 1.5$, $\mu_L = 0$, $Ed = 1$, $cv = 2$, $\underline{u}_0 = 1$, $\underline{u}_1 = 3$, $r = 0.9$, $J_0 = 3$, $J_1 = 1$, $g^+ = 0.04$, $g^- = 0.16$)

customers and different levels of long-term availability β .

When the subcontractor reserves capacity, i.e. when $\beta = 1$, it has to have a high capacity level to guarantee supply all the time. Since the manufacturer will always find an available subcontractor, the backlog level never falls below Z_1 . Therefore the subcontractor is asked to supply either at rate 0 or at rate $\mu_H - \underline{u}_0$. Therefore with $\beta = 1$, the subcontractor has to have a capacity of $C(N) = (\mu_H - \underline{u}_0)N$ to guarantee $\beta = 1$. In the example of Figure 5, when there are 100 producers, the subcontractor has to reserve a capacity of 50 compared to 14 to 19. This shows that the subcontractor benefits considerably from demand pooling.

Alternatively, for a fixed capacity, the subcontractor can determine how many customers can be served with the specified service level. Similarly, for a given number of customers and a given capacity, the subcontractor can determine the maximum production rate $\underline{u}_1$ according to the desired availability.

8. Extensions

8.1 Manufacturer's Capacity Investment Decision

In this section, we investigate the capacity investment decisions of a manufacturer when it is possible to get additional capacity from a subcontractor. A similar problem has been investigated in a queueing setting with a subcontractor that is always available by Bradley (2002a).

Figure 6: Effect of the Demand Variability on Capacity Investments ($\mu_H = 1.5$, $\mu_L = 0$, $Ed = 1$, $\underline{u}_1 = 3$, $\beta = 0.9$, $r = 0.9$, $J_0 = 3$, $J_1 = 1$, $g^+ = 0.04$, $g^- = 0.08$)

For a given maximum production capacity, the manufacturer sets its thresholds, Z_0 and Z_1 and decides how much in-house and subcontracted production to be used to maximize its profit given in (34). Let $\Pi_0(\underline{u}_0)$ be the maximum profit that the manufacturer obtains given that its maximum production rate is $\underline{u}_0$.

Let us assume that that investing in a capacity of $\underline{u}_0$ generates a cost of ζ_0 per unit time. Then the manufacturer's overall profit including the cost of capacity investment is,

$$\Pi'_0 = \Pi_0(\underline{u}_0) - \zeta_0 \underline{u}_0. \quad (39)$$

The manufacturer decides on its optimal capacity level by maximizing Equation (39). A linear cost structure implies an easy way of adjusting capacity. Since the maximization of Equation (39) is performed numerically, a non-linear capacity cost can also be considered.

Figure 6 shows the effect of variability on the manufacturer's capacity investment decisions, and in-house and subcontracted production. Increasing demand variability increases the investment level required by the producer even for the same average demand rate. As the cost of investment increases, more subcontracting is used instead of investing in in-house capacity.

8.2 Subcontractor's Capacity and Pricing Decisions

The subcontractor's profit is determined by the total capacity sold to the manufacturers, its variable production costs, and its fixed costs related to maintaining the capacity required to provide the long-term availability.

Let p_s be the subcontractor's price to a manufacturer for one unit of production. Similarly, let g_s be the unit cost of production for the subcontractor. Furthermore, let us assume that maintaining a capacity of C generates a cost of ζ_1 per unit time for the subcontractor. If the capacity required to provide an availability level of β to N customers is $C(N)$, then the subcontractor's total profit $\underline{\Pi}_1$ is

$$\underline{\Pi}_1 = N\Psi_1(p_s - g_s) - \zeta_1 C(N) \quad (40)$$

Equation (40) exhibits the trade-off problem of the subcontractor. If the subcontractor wants to increase its profit by serving more customers, it needs to increase its capacity $C(N)$ and incur the associated costs. Alternatively, if the subcontractor increases the price p_s , then the manufacturer uses the subcontractor less, i.e., Ψ_1 decreases.

8.2.1 Subcontractor's Capacity Investment Decision

We first investigate the subcontractor's capacity investment decision. Here, we assume that the subcontractor provides the terms of the agreement; p_s (that determines J_1), $\underline{u}_1$, β , and r to each of the N producers. Then each producer maximizes its own profit by determining the optimal values of the hedging levels. The optimal behavior of the producer implies an average subcontracting rate Ψ_1 . Then, the subcontractor maximizes (40) by changing $C(N)$.

8.2.2 Subcontractor's Pricing Decision

We now examine the effects of the subcontractor's pricing decisions. Let the manufacturer's internal unit cost of production be p_c . Obviously, as the cost of subcontracting p_s approaches p_c , each producer uses more subcontracting, i.e., Ψ_1 increases that may require a higher capacity.

We consider two cases. In the first case, the subcontractor sets its price and informs the producer with $\underline{u}_1$, β , and r . Then the manufacturer decides to subcontract optimally. Finally, the subcontractor decides on the capacity level to serve N customers.

Figure 7 depicts the average subcontracting rate of each producer, the required subcontractor's capacity to guarantee β and the subcontractor's total profit, as the ratio of p_s to the manufacturer's internal unit cost of production p_c increases. As p_s increases relative to p_c , subcontracting is not preferred by the producers and therefore the total demand of the subcontractor decreases. The subcontractor can react to this by investing less in its capacity and can maintain a certain level of profit. Note that, even when the costs of in-house and subcontracting are the same, i.e., $p_s = p_c$, the fraction of subcontracted production is limited due to the manufacturer's internal capacity. In other words, the manufacturers try to utilize their own capacity and then consider subcontracting and therefore the fraction of subcontracted production does not reach 100%.

Figure 7 suggests that there is an optimal price of the subcontracting although the profit is quite robust to the changes around the optimal value. This is due to the complex interaction

Figure 7: The case of variable subcontractor capacity ($\mu_H = 1.5$, $\mu_L = 0$, $Ed = 0.9$, $cv = 2$, $\underline{u}_0 = 1$, $\underline{u}_1 = 3$, $p = 0.1$, $r = 0.9$, $J_0 = 5$, $g^+ = 0.08$, $g^- = 0.16$, $g_s = 7$, $\eta_1 = 1$, $N = 50$)

between the manufacturer and the subcontractor. Increasing subcontractor's price reduces the demand for subcontracting from each manufacturer. However, the subcontractor can adjust its capacity and can benefit from serving a given number of customers with a lower capacity.

Figure 7 shows the case where the subcontractor can adjust its own capacity according to the demand. In other words, it represents a case where it is relatively easy to adjust the capacity. In the second case shown in Figure 8, we assume that the subcontractor has a fixed capacity and does not change its capacity dynamically as the demand changes. The negative profit values are the result of the fixed investment costs. In this case, the optimal subcontracting price is more apparent.

Figure 8: The case of fixed total subcontractor's capacity ($\mu_H = 1.5$, $\mu_L = 0$, $Ed = 0.9$, $cv = 2$, $\underline{u}_0 = 1$, $\underline{u}_1 = 3$, $p = 0.1$, $r = 0.9$, $J_0 = 5$, $g^+ = 0.08$, $g^- = 0.16$, $g_s = 7$, $\zeta_1 = 1$, $N = 50$, $C(N) = 20$)

9. Conclusions

In this study, we model and analyze subcontracting with guaranteed availability from the producer's and the subcontractor's perspectives by using a simple model. The model shows that a producer can cope with the risks associated with demand variability by using subcontractors. The costs associated with using a subcontractor can be justified through decreased inventory and backlog levels. The benefit of using subcontractors increases as the demand variability increases. Furthermore, availability of subcontractors affects the producers' investment decisions. Subcontracting can replace capacity investment depending on the investment cost and demand variability.

These conclusions are not affected by having subcontractors that are not available all the time. As opposed to reserving capacity for the producers, subcontractors can guarantee a long term availability. Having a less than 100% availability allows subcontractors benefit from demand pooling. This pooling effect allows subcontractors to have a lower optimal capacity level and incur lower costs. In a competitive environment, these benefits will also be translated into reduced prices to producers.

The model also shows that the producers can react effectively to subcontractors that do not reserve capacity for them by deciding on when to call a subcontractor to provide additional capacity and by deciding on how much safety stock to carry. In other words, the profit they can make is not decreased as a result of this kind of agreement.

This model can be expanded in a number of ways. First, the analysis is performed for a stationary demand for a single product. Extending this model to a multi-product case is very important. Subcontracting can also be very beneficial when a product goes through a product life cycle. A model that investigates the effect of subcontracting in this case is left for future research. Next, at a given time, a subcontractor must decide which customers to serve. This can be modelled as an admission control problem. This extension implies that the time for an unavailable subcontractor to become available is not exponential. Another extension would be allowing a phase-type distribution for this time and analyze the effect of having more reliable supply information.

The model does not incorporate possible fixed costs associated with starting using a subcontractor. Once an agreement is made between the manufacturer and the subcontractor for a period, then there may not be fixed costs. However, in certain manufacturing contexts where a major setup is needed, a fixed start-up cost will be relevant. The model can also be extended in this direction.

Similar to most of the studies in the literature we use expected profit as the main decision factor. Therefore the costs associated with using a subcontractor can be replaced with inventory and backlog costs. However, an additional benefit of subcontractors is decreasing the time to deliver goods that are backordered. The effects of subcontracting on the response time to customers can also be analyzed.

Finally, as an alternative to long-term availability guarantees, a guarantee that gives a maximum time to start production in case the subcontractor is not available can also be used.

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