Addendum to “Unidirectionally invisible potentials as local building blocks of all scattering potentials”

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In [Phys. Rev. A **90**, 023833 (2014)], we offer a solution to the problem of constructing a scattering potential $v(x)$ which possesses scattering properties of one's choice at an arbitrarily prescribed wave number. This solution involves expressing $v(x)$ as the sum of $n \leq 6$ finite-range unidirectionally invisible potentials. We improve this result by reducing the upper bound on n from 6 to 4. In particular, we show that we can construct $v(x)$ as the sum of up to $n = 3$ finite-range unidirectionally invisible potentials, unless if it is required to be bidirectionally reflectionless.

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In Ref. [1], we employ the composition property of the transfer matrix of one-dimensional potential scattering theory to give a complete solution of a local inverse scattering problem that aims at constructing a scattering potential $v(x)$ with given (left and right) reflection and transmission amplitudes, $R^{l/r}$ and T , at a prescribed wave number k_0 . This solution involves expressing $v(x)$ in the form

$$v(x) = v_-(x) + v_1(x) + v_2(x) + v_3(x) + v_4(x) + v_+(x), \quad (1)$$

where $v_{\pm}(x)$ and $v_j(x)$ (with $j = 1, 2, 3, 4$) either vanish or are unidirectionally invisible finite-range potentials with support $I_{\pm}$ and I_j , and transfer matrix $\mathbf{M}_{\pm}$ and $\mathbf{M}_j$, such that

$$I_- \prec I_1 \prec I_2 \prec I_3 \prec I_4 \prec I_+, \quad (2)$$

($I \prec J$ means that I lies on the left of J , i.e., for all $x \in I$ and $y \in J$, $x \leq y$), for $k = k_0$,

$$\mathbf{M}_- = \begin{bmatrix} 1 & 0 \\ -R^l & 1 \end{bmatrix}, \quad \mathbf{M}_+ = \begin{bmatrix} 1 & R^r \\ 0 & 1 \end{bmatrix}, \quad (3)$$

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 \\ \mathcal{R}T & 1 \end{bmatrix}, \quad \mathbf{M}_2 = \begin{bmatrix} 1 & (T-1)/\mathcal{R}T \\ 0 & 1 \end{bmatrix}, \quad (4)$$

$$\mathbf{M}_3 = \begin{bmatrix} 1 & 0 \\ -\mathcal{R} & 1 \end{bmatrix}, \quad \mathbf{M}_4 = \begin{bmatrix} 1 & (1-T)/\mathcal{R} \\ 0 & 1 \end{bmatrix}, \quad (5)$$

and $\mathcal{R}$ is an arbitrary nonzero complex number. The claim that, at $k = k_0$, the reflection and transmission amplitudes of the potential (1) respectively coincide with $R^{l/r}$ and T follows from the fact that the transfer matrix of this potential is given by

$$\mathbf{M} = \mathbf{M}_+ \mathbf{M}_4 \mathbf{M}_3 \mathbf{M}_2 \mathbf{M}_1 \mathbf{M}_- = \begin{bmatrix} T - R^l R^r / T & R^r / T \\ -R^l / T & 1/T \end{bmatrix}. \quad (6)$$

Let $\tilde{v}_1(x)$ and $\tilde{v}_4(x)$ be a pair of finite-range potentials with support $\tilde{I}_1$ and $\tilde{I}_4$, and transfer matrix $\tilde{\mathbf{M}}_1$ and $\tilde{\mathbf{M}}_4$ such that

$\tilde{I}_- \prec \tilde{I}_2, \tilde{I}_3 \prec \tilde{I}_+$, and

$$\tilde{\mathbf{M}}_1 = \mathbf{M}_1 \mathbf{M}_- = \begin{bmatrix} 1 & 0 \\ -R^l + \mathcal{R}T & 1 \end{bmatrix}, \quad (7)$$

$$\tilde{\mathbf{M}}_4 = \mathbf{M}_+ \mathbf{M}_4 = \begin{bmatrix} 1 & R^r + \frac{1-T}{\mathcal{R}} \\ 0 & 1 \end{bmatrix}. \quad (8)$$

Then $\tilde{v}_1(x)$ and $\tilde{v}_4(x)$ are respectively right- and left-invisible potentials, and according to Eq. (6), we have $\mathbf{M} = \tilde{\mathbf{M}}_4 \mathbf{M}_3 \mathbf{M}_2 \tilde{\mathbf{M}}_1$. In light of this relation, for $k = k_0$, the (left and right) reflection and transmission amplitudes of the one-parameter family of potentials

$$v_{\mathcal{R}}(x) := \tilde{v}_1(x) + v_2(x) + v_3(x) + \tilde{v}_4(x), \quad (9)$$

coincide with $R^{l/r}$ and T , respectively.

If $R^l = R^r = 0$, i.e., $v(x)$ is bidirectionally reflectionless, we have $\mathbf{M}_{\pm} = \mathbf{I}$, where $\mathbf{I}$ stands for the 2×2 identity matrix. This shows that we can take $\tilde{v}_1 = v_1$ and $\tilde{v}_4 = v_4$.

If $R^l \neq 0$, we can set $\mathcal{R} = R^l / T$, so that $\tilde{\mathbf{M}}_1 = \mathbf{I}$. This suggests taking $\tilde{v}_1 = 0$, which in turn implies $v_{\mathcal{R}}(x) = v_2(x) + v_3(x) + \tilde{v}_4(x)$ with $\tilde{v}_4$ being a left-reflectionless potential with the right reflection amplitude $R^r + T(1-T)/R^l$ at $k = k_0$. Similarly, for $R^r \neq 0$, the choice $\mathcal{R} = (T-1)/R^r$ allows us to take $\tilde{v}_4 = 0$ and $v_{\mathcal{R}}(x) = \tilde{v}_1(x) + v_2(x) + v_3(x)$, where $\tilde{v}_1$ is a right-reflectionless potential with the left reflection amplitude at $k = k_0$ given by $R^l + T(1-T)/R^r$.

These observations together with the fact that there is an explicitly constructible model for arbitrary unidirectionally invisible finite-range potentials [1] prove the following result.

Theorem. Let k_0 be a positive real number and $\mathcal{R}^l, \mathcal{R}^r$, and $\mathcal{T}$ be arbitrary complex numbers such that $\mathcal{T} \neq 0$. Then there is a finite-range potential $v(x)$ with the following properties:

(1) The reflection and transmission amplitudes of $v(x)$ at $k = k_0$ are respectively given by $R^{l/r} = \mathcal{R}^{l,r}$ and $T = \mathcal{T}$.

(2) If $\mathcal{R}^l = \mathcal{R}^r = 0$ and $\mathcal{T} \neq 1$, $v(x)$ is the sum of four explicitly constructible unidirectionally invisible finite-range potentials with mutually disjoint supports.

(3) If $|\mathcal{R}^l| + |\mathcal{R}^r| \neq 0$, $v(x)$ is the sum of at most three explicitly constructible unidirectionally invisible finite-range potentials with mutually disjoint supports.

(4) The support of $v(x)$ can be chosen to be on the left or the right of any point on the real line.

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This theorem simplifies the application of the local inverse scattering prescription we outline in Ref. [1]. It also underlines the central role unidirectionally invisible potentials play in scattering theory.

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[1] A. Mostafazadeh, *Phys. Rev. A* **90**, 023833 (2014).