Defining sets of full designs with block size three II

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Abstract

A defining set of a t -(v, k, λ) design is a subcollection of its blocks which is contained in a unique t -design with the given parameters. A minimal defining set is a defining set, none of whose proper subcollections is a defining set. The *spectrum* of minimal defining sets of a design D is the set $\{|M| \mid M \text{ is a minimal defining set of } D\}$. The unique simple design with parameters t -($v, k, \binom{v-t}{k-t}$) is said to be the *full design* on v elements. This paper studies the minimal defining sets of full designs when $t = 2$ and $k = 3$. The largest known minimal defining set is given. The existence of a continuous section of the spectrum comprising asymptotically $9v^2/50$ values is shown. This gives a quadratic length section of continuous spectrum where only a linear section with respect to v was known before.

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1 Introduction

A *block design* is a pair $(V, \mathcal{B})$, where V is a v -set of elements and $\mathcal{B}$ is a collection of b k -subsets (called *blocks*) chosen from V such that every element of V occurs in exactly r blocks. We refer to v and b as the *order* and *size* of the design respectively. If every t -subset of V belongs to exactly λ blocks, then we say that the design is *t -balanced*, and we call the block design a *t -design* with parameters $t - (v, k, \lambda)$. When $t = 2$ we refer to this more simply as a (v, k, λ) *design*.

A design is said to be *simple* if it doesn't contain any repeated blocks. The unique simple balanced design with parameters $(v, k, \binom{v-2}{k-2})$ is said to be the *full design on v elements*. It comprises all possible k -tuples on the v -set V , and is therefore the largest possible simple design on v elements.

A set of the blocks which is a subset of an *unique* (v, k, λ) design is said to be a *defining set* of the design, denoted $d(v, k, \lambda)$. A defining set is said to be *minimal* if none of its proper subsets is a defining set, denoted $d_m(v, k, \lambda)$. A defining set is said to be *smallest* if no other defining set of the (v, k, λ) design has smaller cardinality, denoted $d_s(v, k, \lambda)$. The spectrum of a design D is defined $spec(D) = \{|M| \mid M \text{ is a minimal defining set of } D\}$. A continuous portion of the spectrum comprising n values is a set of n consecutive positive integers where for each element of the set there exists at least one minimal defining set with that size.

Let $(V, \mathcal{B})$ be a (v, k, λ) design and suppose that $X \subseteq \mathcal{B}$. Then any (v, k, λ) design on underlying set V which contains the blocks of X is a *completion* of X . Any block which is not in X but is in every completion of X is said to be *forced* (by X). Thus a defining set is a subdesign which has a unique completion, or equivalently one for which every block which is in the design but not the subdesign is forced.

Defining sets of t -designs were first introduced by Ken Gray. He gave preliminary results and lower bounds for the smallest defining sets of some families of these designs in a series of papers [5, 6, 7]. Some theoretical techniques and many different algorithms were suggested for finding smallest and minimal defining sets of combinatorial designs but the minimal defining sets of only a limited number of designs were classified previously [3, 4, 11, 13]. See [10] and [12] for an overview of recent improvements.

In [2] it is shown that every minimal defining set (including each of the smallest ones) of a simple (v, k, λ) design is the intersection of the design with a minimal defining set of the $(v, k, \binom{v-2}{k-2})$ full design. So using the classifications of minimal defining sets of full designs is a comprehensive method for determining the spectrum of all (v, k, λ) designs. Also in [8] it was shown that the sizes of the minimal defining sets of full designs with v elements can be

used to find upper bounds for the number of designs with parameters (v, k, λ) .

We are interested in minimal defining sets of full designs with block size three. From here onwards, we use *full design* to refer to the full design with block size three, and *block* will denote a 3-subset. Let $F(V)$ ($= \{\{x, y, z\} \mid x, y, z \in V, x, y, z \text{ distinct}\}$) denote the full design on the set V with parameters $(v, 3, v-2)$. If D is a defining set of $F(V)$, then $D^C = F(V) \setminus D$.

The term $d(v)$ will denote a defining set of the full design on some v -set. The term $d_m(v)$ will denote a minimal $d(v)$. We are interested in determining the spectrum of allowed sizes for a $d_m(v)$ of $F(V)$, for each $v = |V|$.

A *trade* is a set of blocks T for which there exists a second entirely disjoint set of blocks T' , with the property that every pair of elements in the underlying set occurs equally often in blocks of T and in blocks of T' . We refer to T' as the *disjoint mate* of T . Trades are related to minimal defining sets through the following well-known result, first given in [5]:

Lemma 1.1 *Let D be a subset of the blocks of a design $(V, \mathcal{B})$, and let $D^C = \mathcal{B} \setminus D$. Then*

- (1) *D is a defining set of $(V, \mathcal{B})$ if, and only if, there is no trade $T \subseteq D^C$; that is, every trade contained in $\mathcal{B}$ intersects D .*
- (2) *Suppose that D is a defining set of $(V, \mathcal{B})$. Then D is minimal if, and only if, for every block $B \in D$ there exists a trade T satisfying $T \setminus D^C = \{B\}$.*

The smallest possible trade is known as the *Pasch trade*. For distinct elements u, v, w, x, y, z , the Pasch trade has $T = \{\{u, w, x\}, \{u, y, z\}, \{v, w, y\}, \{v, x, z\}\}$ and $T' = \{\{v, w, x\}, \{v, y, z\}, \{u, w, y\}, \{u, x, z\}\}$. Another trade of which we will make frequent use is given by $T = \{\{t, w, x\}, \{t, v, y\}, \{t, v, z\}, \{u, y, z\}, \{u, v, w\}, \{u, v, x\}\}$, where t, u, v, w, x, y, z are distinct elements. Its disjoint mate is given by $T' = \{\{u, w, x\}, \{u, v, y\}, \{u, v, z\}, \{t, y, z\}, \{t, v, w\}, \{t, v, x\}\}$. We will refer to a trade with this structure as a *Type 2 trade*.

In [1] and [13] constructions are given which produce $d_m(v)$ s of size $(v^3 - 6v^2 + 5v + 12)/6$, $v \geq 6$, and $(v^3 - 6v^2 + 11v - 30)/6$, $v \geq 5$, respectively. These are listed as Constructions 1 and 2 in [2]. A more general form of Construction 2 for block sizes greater than three is given in [9]. In [2] the authors gave more general constructions and developed bounds on the size of $d_m(v)$ s, as summarized by the following lemma.

Lemma 1.2 *Let $v \geq 7$.*

- (1) *There exists a $d_m(v)$ of size n if $\frac{v^3-6v^2+5v+6}{6} \leq n \leq \frac{v^3-6v^2+11v-24}{6}$.*
- (2) *If there exists a $d_m(v)$ of size n , then $\frac{15}{35}\binom{v}{3} \leq n \leq \binom{v}{3} - v$.*

See [2] and [8] for further background and motivation for our study of the defining sets of the full design.

Asymptotically, the established spectrum contains approximately v consecutive values. However, the gap between the highest known value and the upper bound is of order v^2 , while the gap between the smallest known value and the lower bound is of order v^3 .

We improve the known spectrum; our main results are given in Theorems 3.1 and 3.2. The higher end of the spectrum of known values is increased to include, asymptotically, $9v^2/50$ consecutive values. The difference between the size of the largest known $d_m(v)$ and the upper bound is reduced to $2v - 5$ by Lemma 2.2 with $k = 1$.

In Section 2 we give our new constructions, and in Section 3 we calculate the values of the spectrum covered by our constructions.

2 Constructions

We develop some of our constructions with the aid of the following well-known lemma.

Lemma 2.1 *Let U and W be disjoint sets of size u and w respectively. Let D be a $d_m(u)$ of $F(U)$, and let D^* be a set of blocks on $U \cup W$ such that $D^* \cap F(U) = D$. Then*

- (1) *D^* is a $d(u+w)$ of $F(U \cup W)$ if, and only if, all blocks in $(D^*)^C \setminus D^C$ are forced by D^* ; and*
- (2) *D^* is a $d_m(u+w)$ of $F(U \cup W)$ if, and only if, it is a $d(u+w)$ and for every block $B \in D^* \setminus D$ there exists a trade T with $T \setminus (D^*)^C = \{B\}$.*

The following minimal defining set obtained by Construction 1 is the main tool we will use to obtain the results in this paper. We will extend these minimal defining sets by using the extension constructions, Construction 2 and 3 to get minimal defining sets with consecutive sizes for each v .

Construction 1 Let $\{0, 1\}$, $U_1, U_2, \dots, U_k$ and S be disjoint sets with $U_i = \{x_i, y_i, z_i\}$, $1 \leq i \leq k$, and $|S| = s \geq 0$. Let $V = \{0, 1\} \cup \bigcup_{i=1}^k U_i \cup S$, and $v = |V| = 3k + 2 + s$. Define

$$D_1 = F(V) \setminus \left(\bigcup_{i=1}^k F(\{0, 1\} \cup U_i) \cup \bigcup_{\alpha \in S} \{\{0, 1, \alpha\}, \{0, x_1, \alpha\}, \{1, x_1, \alpha\}\} \right).$$

Lemma 2.2 *The set D_1 defined by Construction 1 is a $d_m(v)$ of $F(V)$ of size $(v^3 - 3v^2 - 16v + 36)/6 - k$.*

Proof: The cardinality of D_1 is given by $\binom{v}{3} - k\binom{5}{3} - 3s$. We consider two cases.

Case 1. $s = 0$: To show unique completion, let $C = D_1 \cup R$ be a completion of D_1 to a $(v, 3, v-2)$ design on V ; we will show that $C = F(V)$. If $x \in U_i$ and $y \in U_j$, $i \neq j$, then there is no block in R containing $\{x, y\}$. Thus every block in R is a subset of some $U_i \cup \{0, 1\}$.

Let R_i denote the subset of R comprising blocks in $U_i \cup \{0, 1\}$. For any pair contained in $U_i \cup \{0, 1\}$, other than $\{0, 1\}$, there are exactly three blocks in R_i containing that pair. Thus if $\{0, 1\}$ occurs in n blocks of R_i , then $|R_i| = \frac{9 \cdot 3 + n}{3} = 9 + \frac{n}{3}$.

Suppose that there is some $j \in \{1, 2, \dots, k\}$ such that $|R_j| \neq 10$. Since $\sum_{i=1}^k |R_i| = |R| = |D_1^C| = 10k$, we can assume without loss of generality that $|R_j| < 10$; it follows that $|R_j| = 9$ and $\{0, 1\}$ occurs in no blocks of R_j . However, for each $x \in U_j$, $\{0, x\}$ occurs in three blocks of R_j , implying that there is a total of $9/2$ blocks in R_j which contain 0, a contradiction. Therefore $|R_i| = 10$, for $i = 1, 2, \dots, k$, and each R_i forms a $(5, 3, 3)$ design on $U_i \cup \{0, 1\}$. Since the unique $(5, 3, 3)$ design is the full design of order five, $R = D_1^C$ as required.

To show minimality we need to show that for every $X \in D_1$ there exists a trade T satisfying $T \setminus D_1^C = \{X\}$. Up to generality, we can assume that X is of the form $\{0, x_1, x_2\}$, $\{x_1, y_1, x_2\}$ or $\{x_1, x_2, x_3\}$.

For $X = \{0, x_1, x_2\}$, an appropriate trade is $\{\{0, x_1, x_2\}, \{0, 1, x_2\}, \{0, 1, y_2\}, \{0, 1, z_2\}, \{0, y_1, z_1\}, \{x_2, y_2, z_2\}, \{1, x_1, y_1\}, \{1, x_1, z_1\}\}$, with disjoint mate $\{\{1, x_1, x_2\}, \{0, 1, x_1\}, \{0, 1, y_1\}, \{0, 1, z_1\}, \{1, y_2, z_2\}, \{x_1, y_1, z_1\}, \{0, x_2, y_2\}, \{0, x_2, z_2\}\}$. For $X = \{x_1, y_1, x_2\}$, a suitable (Pasch) trade is: $\{\{x_1, y_1, x_2\}, \{x_1, z_1, 0\}, \{1, y_1, z_1\}, \{1, x_2, 0\}\}$. For $X = \{x_1, x_2, x_3\}$, the required (Type 2) trade is $\{\{x_1, x_2, x_3\}, \{x_1, 0, y_1\}, \{x_1, 0, z_1\}, \{1, y_1, z_1\}, \{1, 0, x_2\}, \{1, 0, x_3\}\}$.

Case 2. $s > 0$: Let $U = \{0, 1\} \cup \bigcup_{i=1}^k U_i$. Note that $D_1 \cap F(U)$ is the $d_m(3k+2)$ of $F(U)$ given by Case 1. Thus, by Lemmas 2.1 and Case 1, when proving unique completion and

minimality, it is only necessary to consider blocks containing at least one element of S .

To show unique completion, let $C = D_1 \cup R$ be a completion of D_1 to a $(v, 3, v - 2)$ design on V . Let α be an arbitrary element of S . The element α occurs with each of the elements $0, 1$ and x_1 in exactly two blocks of D_1^C and hence R . Furthermore, α does not occur with any other elements in blocks of R . It follows immediately that R must contain $\{0, 1, \alpha\}$, $\{0, x_1, \alpha\}$ and $\{1, x_1, \alpha\}$. Thus $R = D_1^C$ as required.

Up to loss of generality, to show minimality we need to find a trade T satisfying $T \setminus D_1^C = \{X\}$ for X of the form $\{\alpha_1, 0, y_1\}$, $\{\alpha_1, 0, x_2\}$, $\{\alpha_1, x_1, y_1\}$, $\{\alpha_1, x_1, x_2\}$, $\{\alpha_1, y_1, z_1\}$, $\{\alpha_1, y_1, x_2\}$, $\{\alpha_1, x_2, y_2\}$, $\{\alpha_1, x_2, x_3\}$, $\{\alpha_1, \alpha_2, 0\}$, $\{\alpha_1, \alpha_2, x_1\}$, $\{\alpha_1, \alpha_2, y_1\}$, $\{\alpha_1, \alpha_2, x_2\}$ or $\{\alpha_1, \alpha_2, \alpha_3\}$, where α_1, α_2 and α_3 are three distinct points in S ; here α_2 and α_3 are only used in the cases where $|S| \geq 2$ and $|S| \geq 3$, respectively.

For $X \in \{\{\alpha_1, 0, y_1\}, \{\alpha_1, x_1, y_1\}, \{\alpha_1, y_1, z_1\}, \{\alpha_1, y_1, x_2\}, \{\alpha_1, x_2, y_2\}, \{\alpha_1, \alpha_2, y_1\}, \{\alpha_1, \alpha_2, x_2\}, \{\alpha_1, \alpha_2, \alpha_3\}\}$ suitable (Pasch) trades are:

$$\begin{aligned} & \{\{\alpha_1, 0, y_1\}, \{x_1, y_1, z_1\}, \{x_1, \alpha_1, 1\}, \{0, z_1, 1\}\}, & \{\{\alpha_1, x_2, y_2\}, \{\alpha_1, 0, 1\}, \{z_2, 0, x_2\}, \{z_2, 1, y_2\}\}, \\ & \{\{\alpha_1, x_1, y_1\}, \{0, y_1, z_1\}, \{0, \alpha_1, 1\}, \{x_1, z_1, 1\}\}, & \{\{\alpha_1, \alpha_2, y_1\}, \{1, x_1, y_1\}, \{0, 1, \alpha_1\}, \{0, x_1, \alpha_2\}\}, \\ & \{\{\alpha_1, y_1, z_1\}, \{0, 1, z_1\}, \{0, x_1, y_1\}, \{1, x_1, \alpha_1\}\}, & \{\{\alpha_1, \alpha_2, x_2\}, \{0, 1, x_2\}, \{0, \alpha_1, x_1\}, \{1, \alpha_2, x_1\}\} \\ & \{\{\alpha_1, y_1, x_2\}, \{\alpha_1, 1, x_1\}, \{0, 1, x_2\}, \{0, x_1, y_1\}\}, & \{\{\alpha_1, \alpha_2, \alpha_3\}, \{\alpha_1, 0, 1\}, \{\alpha_2, 0, x_1\}, \{\alpha_3, 1, x_1\}\}. \end{aligned}$$

For $X \in \{\{\alpha_1, x_1, x_2\}, \{\alpha_1, x_2, x_3\}, \{\alpha_1, \alpha_2, 0\}, \{\alpha_1, \alpha_2, x_1\}\}$ suitable (Type 2) trades are:

$$\begin{aligned} & \{\{x_2, \alpha_1, x_1\}, \{x_2, 0, y_2\}, \{x_2, 0, z_2\}, \{1, y_2, z_2\}, \{1, 0, \alpha_1\}, \{1, 0, x_1\}\}, \\ & \{\{x_2, \alpha_1, x_3\}, \{x_2, 1, y_2\}, \{x_2, 1, z_2\}, \{0, y_2, z_2\}, \{0, 1, \alpha_1\}, \{0, 1, x_3\}\}, \\ & \{\{0, \alpha_1, \alpha_2\}, \{0, 1, y_1\}, \{0, 1, z_1\}, \{x_1, y_1, z_1\}, \{x_1, 1, \alpha_1\}, \{x_1, 1, \alpha_2\}\} \text{ and} \\ & \{\{x_1, \alpha_1, \alpha_2\}, \{x_1, 1, y_1\}, \{x_1, 1, z_1\}, \{0, y_1, z_1\}, \{0, 1, \alpha_1\}, \{0, 1, \alpha_2\}\}. \end{aligned}$$

For $X = \{\alpha_1, 0, x_2\}$ the required trade is $\{\{\alpha_1, 0, x_2\}, \{0, 1, x_2\}, \{0, 1, y_2\}, \{0, 1, z_2\}, \{0, x_1, y_1\}, \{0, x_1, z_1\}, \{1, x_1, \alpha_1\}, \{1, y_1, z_1\}, \{x_2, y_2, z_2\}\}$, with disjoint mate $\{\{\alpha_1, 0, x_1\}, \{0, 1, x_1\}, \{0, 1, y_1\}, \{0, 1, z_1\}, \{0, x_2, y_2\}, \{0, x_2, z_2\}, \{1, x_2, \alpha_1\}, \{1, y_2, z_2\}, \{x_1, y_1, z_1\}\}$.

The following results allow us to extend Construction 1.

Lemma 2.3 [2] *Let U and W be disjoint sets of size $u \geq 5$ and $w \geq 0$ respectively, let D be a $d_m(u)$ of $F(U)$ of size n , and let $a \in U$. Let $V = U \cup W$ and $v = |V|$. Define $D^* = D \cup (F(V \setminus \{a\}) \setminus F(U))$. Then the set D^* is:*

- (1) *a $d(v)$ of $F(V)$ of size $n + \binom{v-1}{3} - \binom{u-1}{3}$; and*
- (2) *minimal if the following condition holds: for every $x_1, x_2 \in U \setminus \{a\}$, and for an arbitrary element $p \notin U$, there exists a trade T with $T \setminus (D^C \cup \{\{p, a, x\} \mid x \in U \setminus \{a\}\}) = \{\{p, x_1, x_2\}\}$.*

Lemma 2.4 [2] *Let U , W and Q be disjoint sets of size $u \geq 5$, $w \geq 2$ and $q \geq 1$, respectively, and let $V = U \cup W \cup Q$ and $v = |V|$. Let $a, b \in U$ and $c \in Q$. Assume that D is a $d_m(u)$ of $F(U)$ of size n , and let $D^C = F(U) \setminus D$. Define*

$$(D^*)^C = D^C \cup \{\{a, y_1, x\}, \{a, y_1, y_2\}, \{a, y_1, c\} \mid x \in U \setminus \{a\}, y_1, y_2 \in W\} \\ \cup \{\{b, z_1, x\}, \{b, z_1, z_2\}, \{b, z_1, y\} \mid x \in U \setminus \{b\}, z_1, z_2 \in Q, y \in W\}.$$

Then $(D^)^C \subseteq F(V)$ and $D^* = F(V) \setminus (D^*)^C$ is*

- (1) *a $d(v)$ of $F(V)$ of size $n + \binom{v-1}{3} - \binom{u-1}{3} - w$; and*
- (2) *minimal if for every $x_1, x_2 \in U \setminus \{\alpha\}$, where $\alpha \in \{a, b\}$, and for an arbitrary element $p \notin U$, there exists a trade T with $T \setminus (D^C \cup \{\{p, \alpha, x\} \mid x \in U \setminus \{\alpha\}\}) = \{\{p, x_1, x_2\}\}$.*

Construction 2 Let $\{0, 1\}$, $U_1, U_2, \dots, U_k$, S and W be disjoint sets with $|U_i| = \{x_i, y_i, z_i\}$, $1 \leq i \leq k$, $|S| = s \geq 0$, and $|W| = w \geq 0$. Define $U = \{0, 1\} \cup \bigcup_{i=1}^k U_i \cup S$, $u = |U| = 3k + 2 + s$, $V = U \cup W$, and $v = |V| = u + w$. Note that $v \geq 5$.

Let D_1 be as defined in Construction 1, but on underlying set U rather than V . Define

$$D_2 = D_1 \cup (F(V \setminus \{0\}) \setminus F(U)).$$

Construction 3 Let $\{0, 1\}$, $U_1, U_2, \dots, U_k$, S , W and Q be disjoint sets with $|U_i| = \{x_i, y_i, z_i\}$, $1 \leq i \leq k$, $|S| = s \geq 0$, $|W| = w' \geq 2$ and $|Q| = q \geq 1$. Let $c \in Q$ be a distinguished point, and let $w = w' + q$. Define $U = \{0, 1\} \cup \bigcup_{i=1}^k U_i \cup S$, $u = |U| = 3k + 2 + s$, $V = U \cup W \cup Q$, and $v = |V| = u + w$. Note that $v \geq 8$.

Let D_1 be as defined in Construction 1, but on underlying set U rather than V . Define

$$D_3 = D_1 \cup (F(V) \setminus (F(U) \cup \{\{0, \beta_1, \alpha\}, \{0, \beta_1, \beta_2\}, \{0, \beta_1, c\} \mid \alpha \in U \setminus \{0\}, \beta_1, \beta_2 \in W\} \\ \cup \{\{1, \gamma_1, \alpha\}, \{1, \gamma_1, \gamma_2\}, \{1, \gamma_1, \beta\} \mid \alpha \in U \setminus \{1\}, \gamma_1, \gamma_2 \in Q, \beta \in W\}))).$$

Lemma 2.5 *The sets D_2 and D_3 defined by Constructions 2 and 3 are $d_m(v)$ s of sizes*

$$\frac{v^3 - 3v^2 - 16v + 36}{6} - \frac{w(2v - 9 - w)}{2} - k \text{ and } \\ \frac{v^3 - 3v^2 - 16v + 36}{6} - \frac{w(2v - 9 - w)}{2} - w' - k, \text{ respectively.}$$

Proof.

By Lemma 2.2, the set D_1 used in Constructions 2 and 3 is a $d_m(u)$ of size $(u^3 - 3u^2 - 16u + 36)/6 - k$. The sizes of D_2 and D_3 follow by Lemmas 2.3 and 2.4 respectively (or by direct count). By Lemma 2.3 (with D , D^* and a corresponding to D_1 , D_2 and 0 respectively), to prove that D_2 is a $d_m(v)$ it is only necessary to find, for an arbitrary $p \notin U$ and for every $\alpha_1, \alpha_2 \in U \setminus \{0\}$, a trade T satisfying $T \setminus (D_1^C \cup \{\{p, 0, \alpha\} \mid \alpha \in U \setminus \{0\}\}) = \{\{p, \alpha_1, \alpha_2\}\}$. We give these trades below. By symmetry, we can substitute 1 for 0 in this condition; thus by Lemma 2.4 (with D , D^* a and b corresponding to D_1 , D_3 , 0 and 1 respectively), it follows that D_3 is also a $d_m(v)$.

We begin by considering the case where neither point is in S ; that is, $\alpha_1, \alpha_2 \in \{1\} \cup \bigcup_{i=1}^k U_i$. Since the following trades do not use any elements of S , we can regard U_i and U_j as interchangeable, and x_i, y_i, z_i as interchangeable, for every i and j . Thus up to generality we have $\{\alpha_1, \alpha_2\} \in \{\{1, x_1\}, \{x_1, y_1\}, \{x_1, x_2\}\}$. Suitable (Pasch) trades are $\{\{p, 1, x_1\}, \{p, 0, y_1\}, \{0, 1, z_1\}, \{x_1, y_1, z_1\}\}$, $\{\{p, x_1, y_1\}, \{p, 0, 1\}, \{0, x_1, z_1\}, \{1, y_1, z_1\}\}$ and $\{\{p, x_1, x_2\}, \{p, 0, y_1\}, \{1, 0, x_2\}, \{1, x_1, y_1\}\}$.

If $\alpha_1, \alpha_2 \in S$, the required (Pasch) trade is: $\{\{p, \alpha_1, \alpha_2\}, \{p, 0, 1\}, \{\alpha_1, 0, x_1\}, \{\alpha_2, 1, x_1\}\}$.

We are left with the case $\alpha_1 \in S, \alpha_2 \notin S$. Up to generality we have α_2 equal to 1, x_1, y_1 or x_2 . The required (Pasch) trades are $\{\{p, \alpha_1, 1\}, \{p, 0, y_1\}, \{0, x_1, \alpha_1\}, \{1, x_1, y_1\}\}$, $\{\{p, \alpha_1, x_1\}, \{p, 0, y_1\}, \{0, 1, \alpha_1\}, \{1, x_1, y_1\}\}$, $\{\{p, \alpha_1, y_1\}, \{p, 0, x_1\}, \{0, 1, \alpha_1\}, \{1, x_1, y_1\}\}$ and $\{\{p, \alpha_1, x_2\}, \{p, 0, y_2\}, \{0, 1, \alpha_1\}, \{1, x_2, y_2\}\}$.

3 Evaluating the spectrum

We now calculate, for a given v , the range of $d_m(v)$ sizes given by Constructions 2 and 3.

Theorem 3.1 *For any integers v, w satisfying $0 \leq w \leq v - 5$, let*

$$\Omega_{v,w} = \frac{v^3 - 3v^2 - 16v + 30}{6} - \frac{w(2v - 9 - w)}{2} \quad \text{and} \quad \Delta_{v,w} = \left\lfloor \frac{v + 2w - 8}{3} \right\rfloor.$$

Then there exists a $d_m(v)$ of size n provided that $\Omega_{v,w} - \Delta_{v,w} \leq n \leq \Omega_{v,w}$, except that if $w \geq v - 7$ then there may not be a $d_m(v)$ of size $\Omega_{v,w} - 1$. In addition, there is a $d_m(v)$ of size $\Omega_{v,0} - \left\lfloor \frac{v-5}{3} \right\rfloor = \Omega_{v,0} - \Delta_{v,0} - 1$.

Proof. In Constructions 2 and 3, we have $v = 3k + 2 + s + w$, where $k \geq 1, s \geq 0$ and $w \geq 0$, and in Construction 3 we also have $2 \leq w' \leq w - 1$ (implying that $w \geq 3$). Thus

for a fixed v and w , which must satisfy $0 \leq w \leq v - 5$, k and w' can take any integer value satisfying $1 \leq k \leq (v - w - 2)/3$ and $2 \leq w' \leq w - 1$. The result follows from Lemma 2.5.

Note that Construction 2 gives a defining set of size equivalent to the defining set given by Construction 3 with $w' = 0$. The fact that $w' \neq 1$ only matters when there is no flexibility in choosing k ; that is, the upper bound on k is less than 2, or equivalently $w \geq v - 7$. In this case neither construction can be of size $\Omega_{v,w} - 1$.

Theorem 3.2 *Let $v \geq 12$, and let $r \equiv -2v \pmod{5}$, with $0 \leq r < 5$. Then there exists a $d_m(v)$ of size n provided that*

$$\frac{v^3 - 6v^2 + 5v + 6}{6} \leq n \leq \frac{25v^3 - 123v^2 + v(50 - 18r) + (3r^2 + 75r - 300)}{150}.$$

Proof. Let w take any integer value in the range $0 \leq w \leq v - 5$, and let $\Omega_{v,w}$ and $\Delta_{v,w}$ be as defined in Theorem 3.1. Note that $\Omega_{v,w}$ is a decreasing function of w . Consider the range of $d_m(v)$ sizes given by Theorem 3.1 for each w . The smallest size when $w = m$ (namely $\Omega_{v,m} - \Delta_{v,m}$) is at most one more than the largest size when $w = m + 1$ (namely $\Omega_{v,m+1}$) if and only if $-\frac{m(2v-9-m)}{2} - \lfloor \frac{v+2m-8}{3} \rfloor \leq -\frac{(m+1)(2v-9-(m+1))}{2} + 1$; that is, $m \geq w_0$, where $w_0 = \lceil \frac{2v-10}{5} \rceil$. Thus, by Theorem 3.1, we have a $d_m(v)$ of every size between $\Omega_{v,v-5} - \Delta_{v,v-5}$ and Ω_{v,w_0} , except that $\Omega_{v,w} - 1$ is a possible exception for $w \in \{v - 5, v - 6, v - 7\}$.

Any possible exceptions occur in the range between $\Omega_{v,v-5} - \Delta_{v,v-5}$ and $\Omega_{v,v-8}$. To deal with these possible exceptions, we repeat the calculation above but with a tighter condition, for $m \in \{v - 6, v - 7, v - 8\}$. For $w \geq v - 7$, we have $\Delta_{v,w} \geq 2$ (using $v \geq 12$). Thus for these values of w Theorem 3.1 still gives one or more consecutive sizes, from $\Omega_{v,w} - \Delta_{v,w}$ to $\Omega_{v,w} - 2$. Restricting our attention to this range, the smallest cardinality with $w = m$ (namely $\Omega_{v,m} - \Delta_{v,m}$) is at most one more than the largest cardinality with $w = m + 1$ (namely $\Omega_{v,m+1} - 2$) if $-\frac{m(2v-9-m)}{2} - \lfloor \frac{v+2m-8}{3} \rfloor \leq -\frac{(m+1)(2v-9-(m+1))}{2} - 1$; that is, $m \geq \lceil \frac{2v-4}{5} \rceil$. This condition holds for all $v \geq 12$ and $m \geq v - 8$. Thus we have a $d_m(v)$ of every size between $\Omega_{v,v-5} - \Delta_{v,v-5}$ and $\Omega_{v,v-8}$, which implies that we have dealt with the possible exceptions, and so in fact we have a $d_m(v)$ of every size between $\Omega_{v,v-5} - \Delta_{v,v-5}$ and Ω_{v,w_0} . The result follows by Theorem 3.1 and the evaluation of Ω_{v,w_0} .

4 Concluding Remarks

The largest construction we know for a general v is Construction 2 with $w = 0$ and $k = 1$; it has size $(v^3 - 3v^2 - 16v + 30)/6$. By Theorem 3.1 (with $w = 0$), we can in fact construct

a $d_m(v)$ of each of the $\lfloor \frac{v-2}{3} \rfloor$ consecutive sizes up to this value. At the opposite end of the spectrum, Theorem 3.1 with $w = v - 5$ gives $v - 7$ consecutive sizes, with the smallest being $(v^3 - 6v^2 + 5v^2 + 6)/6$. This is more or less the result given in [2]. But Theorem 3.2 significantly improves this result when $v \geq 12$; asymptotically, [2] gives approximately v consecutive sizes, but Theorem 3.2 shows that Constructions 2 and 3 give approximately $9v^2/50$ consecutive sizes. This is (asymptotically) over one third of the range between the smallest and largest minimal defining sets known.

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