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Citation: [Journal of Mathematical Physics](#) **54**, 041502 (2013); doi: 10.1063/1.4798786

View online: <http://dx.doi.org/10.1063/1.4798786>

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Global solvability and blow up for the convective Cahn-Hilliard equations with concave potentials

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(Received 21 January 2013; accepted 14 March 2013; published online 4 April 2013)

We study initial boundary value problems for the unstable convective Cahn-Hilliard (CH) equation, i.e., the Cahn Hilliard equation whose energy integral is not bounded below. It is well-known that without the convective term, the solutions of the unstable CH equation $\partial_t u + \partial_x^4 u + \partial_x^2(|u|^p u) = 0$ may blow up in finite time for any $p > 0$. In contrast to that, we show that the presence of the convective term $u \partial_x u$ in the Cahn-Hilliard equation prevents blow up at least for $0 < p < \frac{4}{9}$. We also show that the blowing up solutions still exist if p is large enough ($p \geq 2$). The related equations like Kolmogorov-Sivashinsky-Spiegel equation, sixth order convective Cahn-Hilliard equation, are also considered. © 2013 American Institute of Physics. [<http://dx.doi.org/10.1063/1.4798786>]

I. INTRODUCTION

It is well-known that the solutions of the semilinear heat equations with concave potentials

$$\partial_t u - \Delta u - u|u|^p = 0$$

blow up in finite time if $p > 0$ and the initial energy is negative, see, e.g., Refs. 1, 16, 18, 19, and 26 and references therein. However, it is also established that the presence of the *convective* terms in the semilinear parabolic equation prevents blow up if the nonlinear source term is not growing very rapidly. For instance, the solutions of the following convective heat equation

$$\partial_t u + u \partial_x u - \partial_x^2 u - u|u|^p = 0 \quad (1.1)$$

in a bounded interval $\Omega = [-L, L]$ with Dirichlet boundary conditions exist globally in time if $p \leq 1$ and the blowing up solutions occur only if $p > 1$, see Refs. 5, 20, 21, and 28, see also Ref. 30 for the results on suppressing the blow up by adding the sufficiently large linear convective terms in reaction-diffusion equations.

The main aim of the present paper is to study the analogous problems for the following convective Cahn-Hilliard (CH) equation with concave potentials

$$\partial_t u + \partial_x^2(\partial_x^2 u + u|u|^p) + u \partial_x u = 0 \quad (1.2)$$

in a bounded segment $\Omega = [-L, L]$ endowed by *periodic* boundary conditions.

The long-time behavior of solutions of initial boundary value problems for CH and relevant equations are intensively studied by many authors, see Refs. 2, 3, 9, 13, 15, 24, and 29 and references therein. For instance, the existence of the blowing up solutions for Eq. (1.2) *without* the convective term $u \partial_x u$ is known for any $p > 0$, see, e.g., Refs. 9, 10, 22, and 25. However, based on the reaction-diffusion experience mentioned above, one may expect that the convective term prevents blow-up here as well. We will show below that this is indeed the case. Namely, the following theorem can be considered as the main result of the paper.

Theorem 1.1. *Let $0 < p < \frac{4}{9}$. Then, for every initial data $u_0 \in L^2(\Omega)$ with zero mean, problem (1.2) possesses a unique solution $u(t)$ which exists for all $t \geq 0$ and remains bounded when $t \rightarrow \infty$.*

Let $p \geq 2$. Then, there are initial data $u_0 \in L^2(\Omega)$ with zero mean such that the corresponding solution blows up in finite time.

Note that, in contrast to the situation with the semilinear heat equations, the result of Theorem 1.1 is not complete in the sense that we know nothing about the behavior of solutions for $\frac{4}{9} \leq p < 2$. Indeed, the sharp result for the semilinear heat equations stated above is strongly based on the maximum principle which we do not have for the CH equations, so we have to use alternative less powerful methods. We hope to return to this problem somewhere else.

The paper is organized as follows.

In Sec. II, we introduce the notations and main technical tools which are necessary for our proof of Theorem 1.1.

Section III is devoted to the proof of the blow up for the case $p \geq 2$. Actually, in the non-convective case, sufficient conditions of blow up of solutions for the CH equation can be established by concavity method of Levine (see Ref. 19) rewriting the equation in the form of a nonlinear differential operator equations of the form

$$Pu_t + Au = F(u) \quad (1.3)$$

in a Hilbert space (see Ref. 15). Here P, A are positive self-adjoint operators and $F(\cdot)$ is a nonlinear gradient operator. But the convective CH equation (1.2) cannot be written in the form (1.3), so the method does not work directly and its adaptation to our situation requires some more delicate arguments and works only under the extra assumption $p \geq 2$.

In Sec. IV, we prove the global existence and dissipativity of solutions of (1.2) in the case $0 < p < \frac{4}{9}$. Our proof uses the so-called Goodmann trick (see Ref. 14) which is nowadays the standard (for the theory of Kuramoto-Sivashinsky equation) method to “extract” the dissipation from the convective term. However, in order to compensate the concave term $u|u|^p$, we need to choose the auxiliary function in this method depending on the size of the initial data and then use the so-called Gronwall lemma with parameter (see Refs. 12 and also 31, where this lemma was implicitly used to establish the global existence of solutions for the Navier-Stokes equations in a strip). Note that our approach also essentially employs the improved construction of the auxiliary function given in Ref. 4.

Finally, in Sec. V, we consider some related problems which can be straightforwardly solved using the methods developed in the paper. In particular, we study here the problem of obtaining uniform in $\delta \rightarrow 0$ upper bounds for the attractor of the so-called Kolmogorov-Sivashinsky-Spiegel equation

$$\partial_t u + \partial_x^4 u + \partial_x^2(2u - \delta u^3) + u \partial_x u = 0. \quad (1.4)$$

Furthermore, the so-called sixth order convective Cahn-Hilliard equation

$$\partial_t u - \partial_x^4(\partial_x^2 u + u - u^3) + u \partial_x u = 0 \quad (1.5)$$

is considered there.

II. NOTATIONS AND PRELIMINARIES

In this section, we introduce the notations which will be used throughout the paper and introduce some technical tools important for what follows.

As usual, we denote by $H^m = H^m(-L, L)$ the Sobolev space of distributions whose derivatives up to order m belong to $L^2(-L, L)$. We write $H = H(-L, L)$ instead of $H^0 = L^2(-L, L)$ and $(\cdot, \cdot)$ stands for the usual scalar product in the Hilbert space H .

The closure of $C_0^\infty(-L, L)$ in $H^m(-L, L)$ will be denoted by $H_0^m = H_0^m(-L, L)$ and $H_{per}^m = H_{per}^m(-L, L)$ stands for the subspace of H^m which consists of $2L$ periodic functions. This definition works only for $m \in \mathbb{N}$, for negative or/and fractional m s, the corresponding Sobolev spaces are defined in a standard way using the duality and interpolation arguments, respectively, see, e.g., Ref. 29.

For every $u \in H_{per}^m$, $m \in \mathbb{N}$, we introduce the mean value operator

$$\langle u \rangle := \frac{1}{2L} \int_{-L}^L u(x) dx$$

and denote by $\dot{H}_{per}^m = \dot{H}_{per}^m(-L, L)$ the subspace of H_{per}^m which consists of functions with zero mean:

$$\dot{H}_{per}^m := \{u \in H_{per}^m, \langle u \rangle = 0\}.$$

We also introduce the inverse of the Laplace operator $P := (-\partial_x^2)^{-1}$ defined on the functions with zero mean. It is well known that this operator gives the isomorphism between the spaces $\dot{H}_{per}^m$ and $\dot{H}_{per}^{m+2}$ for every $m \in \mathbb{N}$:

$$P : \dot{H}_{per}^m \rightarrow \dot{H}_{per}^{m+2}, \quad P \dot{H}_{per}^m = \dot{H}_{per}^{m+2}$$

and the following relations hold for all $u \in \dot{H}^1$:

$$\|u\|_H \leq \bar{d}_0 \|\partial_x u\|_H, \quad \|P^{\frac{1}{2}}(\partial_x u)\|_H = \|u\|_H, \quad \bar{d}_0 := \frac{L}{\pi}. \quad (2.1)$$

Applying the operator P to the both sides of Eq. (1.2), we rewrite it in the equivalent, but more convenient (for our purposes) form

$$P \partial_t u - \partial_x^2 u - P(u \partial_x u) = u|u|^p - \langle u|u|^p \rangle. \quad (2.2)$$

We say that a function $u(t, x)$ is a weak solution of (1.2) (or equivalently of (2.2)) on the time interval $t \in [0, T]$ if

$$u \in C([0, T], \dot{H}_{per}) \cap L^2([0, T], \dot{H}_{per}^2), \quad u \in L^{p+1}([0, T], L^{p+1}(-L, L)) \quad (2.3)$$

and Eq. (2.2) is satisfied in the sense of distributions.

The following theorem gives the *local* well-posedness of this problem.

Theorem 2.1. *Let $0 < p < 4$. Then, for every $u_0 \in \dot{H}_{per}$, the problem (2.2) possesses a unique weak solution defined on the time interval $t \in [0, T]$, where $T = T(u_0) > 0$ depends on the H -norm of the initial data u_0*

The proof of this theorem is standard and, by this reason, is omitted.

Remark 2.2. As follows from the embedding theorem,

$$u \in C([0, T], \dot{H}_{per}) \cap L^2([0, T], \dot{H}_{per}^2)$$

implies that $u \in L^{10}([0, T], L^{10})$. Therefore, the assumption $p < 4$ guarantees that the nonlinear term $u|u|^p$ belongs to $L^2([0, T], H)$ and therefore it is subordinated to the linear terms in the equation.

Let us mention also that the usual parabolic smoothing property works for such weak local solutions, so the factual smoothness of the solution $u(t)$ for $t > 0$ is restricted only by the smoothness of the nonlinearity $u|u|^p$ at $u = 0$. In particular, since this nonlinearity is of at least $C^{1+\alpha}$ for some $\alpha > 0$ (depending on p), one can show that we have at least $u(t) \in C^3(-L, L)$ for $t \in (0, T]$ and any weak solution constructed in Theorem 2.1. Since this regularity is more than enough to justify all estimates used in the paper, we will not return to the questions of *local* well-posedness in what follows and will only concentrate ourselves on derivation of the *a priori* estimates which guarantee the *global* well-posedness or finite-time blow up of solutions.

In the case $p \geq 4$, one can obtain the similar local well-posedness result just using more regular solutions, say,

$$u \in C([0, T], \dot{H}_{per}^1) \cap L^2([0, T], \dot{H}_{per}^3)$$

and starting from more regular initial data $u_0 \in \dot{H}_{per}^1$.

We conclude this section by stating two crucial lemmas, one of them will allow us to find sufficient conditions for blow up of solutions of the CH equation (1.2) and the other one will give the part of Theorem 1.1 related with the global solvability.

Lemma 2.3. (19) Let $\Psi(t)$ be twice continuously differentiable function that satisfies the inequality

$$\Psi''(t)\Psi(t) - (1 + \alpha)[\Psi(t)]^2 \geq 0, \quad t > 0, \quad (2.4)$$

and

$$\Psi(0) > 0, \quad \Psi'(0) > 0, \quad (2.5)$$

where $\alpha > 0$ is a given number. Then there exists

$$t_1 \leq T_1 = \frac{\Psi(0)}{\alpha \Psi'(0)}$$

such that

$$\Psi(t) \rightarrow \infty \quad \text{as } t \rightarrow t_1^-.$$

Lemma 2.4. (12 and 23) Suppose that $\alpha > \beta \geq 1$ and $\gamma \geq 0$ are given numbers that satisfy the inequality,

$$\frac{\beta - 1}{\alpha - 1} < \frac{1}{\gamma + 1}, \quad (2.6)$$

Suppose also that Ψ is a non-negative absolutely continuous function on $[0, \infty)$ which satisfies, for some numbers $K \geq 0$, $\varepsilon_0 > 0$, $M > 0$ and for every $\varepsilon \in (0, \varepsilon_0)$ the differential inequality

$$\Psi'(t) + \varepsilon \Psi(t) \leq K \varepsilon^\alpha [\Psi(t)]^\beta + M \varepsilon^{-\gamma} \quad (2.7)$$

Then there exists a monotone function $Q : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and a positive number $\kappa > 0$ such that

$$\Psi(t) \leq Q(\Psi(0))e^{-\kappa t} + Q(M). \quad (2.8)$$

Moreover, the dissipative estimate (2.8) remains true if the function $\Psi \geq 0$ is only continuous and satisfies the integrated version of inequality (2.7)

$$\Psi(t) \leq \Psi(0)e^{-\varepsilon t} + K \varepsilon^\alpha \int_0^t e^{-\varepsilon(t-s)} [\Psi(s)]^\beta ds + M \varepsilon^{-\gamma-1} \quad (2.9)$$

for all $t \geq 0$.

III. BLOW UP OF SOLUTIONS TO CONVECTIVE CH EQUATIONS

In this section we consider the following problem in $\Omega := [-L, L]$:

$$\begin{cases} \partial_t u + \partial_x^2(\partial_x^2 u + u^3) + u \partial_x u = 0, \\ u|_{t=0} = u_0 \end{cases} \quad (3.1)$$

endowed by periodic boundary conditions. According to Theorem 2.1, this problem has a unique local weak solution for all $u_0 \in \dot{H}_{per}$ and it is equivalent to the following one:

$$\begin{cases} P \partial_t u - \partial_x^2 u + P(u \partial_x u) = u^3 - \langle u^3 \rangle, \\ u|_{t=0} = u_0, \end{cases} \quad (3.2)$$

see Sec. II.

Our aim is to show that for some class of initial functions the corresponding solutions blow up in a finite time. For simplicity, we restrict ourselves to consider only the case $p = 2$ in Eq. (1.2), although as it is not difficult to see, the similar arguments work for all $p \geq 2$.

The main result of the section is the following theorem.

Theorem 3.1. Suppose that u is a solution of the problem (3.2) corresponding to the initial data $u_0 \in \dot{H}_{per}$ which is not equal zero identically and

$$E_0 := -\frac{\lambda}{2} \|P^{1/2} u_0\|^2 - \frac{1}{2} \|\partial_x u_0\|^2 + \frac{1}{4} (u_0^4, 1) \geq 0, \quad (3.3)$$

where $\lambda = \frac{1}{2} + 3d_0^2$ and $d_0 := \max\{\bar{d}_0, 1\}$, see (2.1).

Then there exists $t_1 < \infty$ such that

$$\|P^{1/2} u(t)\| \rightarrow \infty, \text{ as } t \rightarrow t_1^-.$$

Proof. We make the change $u = e^{\lambda t} v$. Then the function v solves the problem

$$\begin{cases} P \partial_t v + \lambda P v - \partial_x^2 v + e^{\lambda t} P(v \partial_x v) = e^{2\lambda t} (v^3 - \langle v^3 \rangle), \\ v|_{t=0} = u_0. \end{cases} \quad (3.4)$$

Let us consider the function

$$\Psi(t) := \int_0^t \|P^{1/2} v(\tau)\|^2 d\tau + C_0, \quad (3.5)$$

where C_0 is some positive parameter to be chosen below. Clearly

$$\Psi'(t) = \|P^{1/2} v(t)\|^2 = 2 \int_0^t (P \partial_\tau v, v) d\tau + \|P^{1/2} u_0\|^2. \quad (3.6)$$

Employing Eq. (3.4) we also obtain that

$$\Psi''(t) = 2(P \partial_t v, v) = -2\lambda \|P^{1/2} v\|^2 - 2\|\partial_x v\|^2 - 2e^{\lambda t} (P(v \partial_x v), v) + 2e^{2\lambda t} (v^4, 1). \quad (3.7)$$

By using the Cauchy inequality with ε and the Schwarz inequality we obtain

$$\begin{aligned} 2e^{\lambda t} |(P(v \partial_x v), v)| &= 2e^{\lambda t} |(P^{1/2}(v \partial_x v), P^{1/2} v)| \leq 2e^{\lambda t} \|P^{1/2}(v \partial_x v)\| \|P^{1/2} v\| \leq \\ &\leq 2d_0 e^{\lambda t} \|v\|^2 \|P^{1/2} v\| \leq \varepsilon_1 e^{2\lambda t} (v^4, 1) + \frac{d_0^2}{\varepsilon_1} \|P^{1/2} v\|^2. \end{aligned}$$

Thus (3.7) implies

$$\Psi''(t) \geq -\left(2\lambda + \frac{d_0^2}{\varepsilon_1}\right) \|P^{1/2} v\|^2 - 2\|\partial_x v\|^2 + (2 - \varepsilon_1) e^{2\lambda t} (v^4, 1). \quad (3.8)$$

Multiplying Eq. (3.4) by $\partial_t v$ and integrating over $(-L, L)$ we obtain the second main energy equality:

$$\frac{d}{dt} E(t) = \|P^{1/2} \partial_t v\|^2 + \frac{\lambda}{2} e^{2\lambda t} (v^4, 1) + e^{\lambda t} (P(v \partial_x v), \partial_t v), \quad (3.9)$$

where

$$E(t) := -\frac{\lambda}{2} \|P^{1/2} v\|^2 - \frac{1}{2} \|\partial_x v\|^2 + \frac{1}{4} e^{2\lambda t} (v^4, 1).$$

Let us estimate the last term on the right-hand side of (3.9):

$$e^{\lambda t} |(P(v \partial_x v), \partial_t v)| \leq e^{\lambda t} d_0 \|v\|^2 \|P^{1/2} \partial_t v\| \leq \varepsilon_2 \|P^{1/2} \partial_t v\|^2 + \frac{d_0^2}{4\varepsilon_2} e^{2\lambda t} (v^4, 1). \quad (3.10)$$

If

$$\lambda \geq \frac{d_0^2}{2\varepsilon_2} \quad (3.11)$$

then it follows from (3.9) and (3.10) that

$$\frac{d}{dt} E(t) \geq (1 - \varepsilon_2) \|P^{1/2} \partial_t v\|^2.$$

Hence

$$E(t) \geq (1 - \varepsilon_2) \int_0^t \|P^{1/2} \partial_\tau v(\tau)\|^2 d\tau + E_0. \quad (3.12)$$

It follows from (3.8) that

$$\Psi''(t) \geq 4(2 - \varepsilon_1)E(t) + \left[2(1 - \varepsilon_1)\lambda - \frac{d_0^2}{\varepsilon_1}\right] \|P^{1/2} v\|^2 + (2 - 2\varepsilon_1) \|\partial_x v\|^2.$$

Let us take $\varepsilon_1 = \frac{1}{2}$ (it is important to have $4(2 - \varepsilon_1) > 4$). Then we have the following estimate for $\Psi''(t)$:

$$\Psi''(t) \geq 6E(t) + (\lambda - 2d_0^2)\Psi'(t) + \|\partial_x v\|^2. \quad (3.13)$$

Due to (3.12) and (3.13) with $\varepsilon_2 = \frac{1}{6}$ (note that for this ε_2 the inequality (3.11) is satisfied) we have

$$\Psi''(t) \geq 5 \int_0^t \|P^{1/2} \partial_\tau v(\tau)\|^2 d\tau + (\lambda - 2d_0^2)\Psi'(t). \quad (3.14)$$

Here we have used the condition $E(0) = E_0 \geq 0$. By using (3.6) and (3.14) we obtain the following inequality

$$\begin{aligned} \Psi''(t)\Psi(t) - \frac{5}{4}(\Psi'(t))^2 &\geq 5 \left(\int_0^t \|P^{1/2} \partial_\tau v\|^2 d\tau \right) \left(\int_0^t \|P^{1/2} v\|^2 d\tau + C_0 \right) + \\ &\quad - 5 \left(\int_0^t (P^{1/2} \partial_\tau v, P^{1/2} v) d\tau + \frac{1}{2} \|P^{1/2} u_0\|^2 \right)^2 + (\lambda - 2d_0^2)\Psi\Psi' = \\ &= 5 \left[\left(\int_0^t \|P^{1/2} \partial_\tau v\|^2 d\tau + C_0 \right) \Psi(t) - \left(\int_0^t (P^{1/2} \partial_\tau v, P^{1/2} v) d\tau + \frac{\|P^{1/2} u_0\|^2}{2} \right)^2 \right] + \\ &\quad (\lambda - 2d_0^2)\Psi(t)\Psi'(t) - C_0\Psi(t). \end{aligned} \quad (3.15)$$

Let us choose in (3.15)

$$C_0 = \frac{1}{2} \|P^{1/2} u_0\|^2.$$

Then the expression in the square brackets is non-negative due to Cauchy-Schwartz inequality. Therefore we have

$$\Psi''(t)\Psi(t) - \frac{5}{4}(\Psi'(t))^2 \geq (\lambda - 2d_0^2)\Psi(t)\Psi'(t) - C_0\Psi(t). \quad (3.16)$$

According to (3.13) the function $\Psi'(t)$ is a non-decreasing function. Thus

$$\Psi'(t) = \|P^{1/2} v(t)\|^2 \geq \|P^{1/2} u_0\|^2.$$

Hence we obtain from (3.16)

$$\Psi''(t)\Psi(t) - \frac{5}{4}[\Psi'(t)]^2 \geq (\lambda - 2d_0^2)\Psi(t)\|P^{1/2} u_0\|^2 - C_0\Psi(t) \geq [2(\lambda - 2d_0^2) - 1]C_0\Psi(t).$$

Finally noting that $\lambda = \frac{1}{2} + 3d_0^2$ we obtain:

$$\Psi''(t)\Psi(t) - \frac{5}{4}[\Psi'(t)]^2 \geq 0.$$

Hence due to Lemma 2.3 the statement of Theorem 3.1 holds true. $\square$

Remark 3.2. It is clear that Theorem 3.1 is true for solutions of Eq. (3.1) under the homogeneous Dirichlet's boundary conditions.

Remark 3.3. It is easy to see that the result of Theorem 3.1 remains true also for the multi-dimensional convective CH equation of the form

$$\partial_t u + \Delta(\Delta u + u^3) + u \vec{b} \cdot \nabla u = 0,$$

where $\vec{b} \in L^\infty(\Omega, \mathbb{R})$ is a given vector field and u satisfies the appropriate boundary conditions.

IV. THE CONVECTIVE CH EQUATIONS: GLOBAL EXISTENCE

In this section, we continue our study of the convective CH equation:

$$\partial_t u + u \partial_x u + \partial_x^2(\partial_x^2 u + u|u|^p) = 0 \quad (4.1)$$

on the interval $\Omega = [-1, 1]$ (for simplicity, we take $L = 1$ here) endowed with the periodic conditions. We have seen in Sec. III that this equation possesses the blowing up in finite time solutions if $p \geq 2$. The aim of the present section is to show that the presence of the convective term *prevents* the blow up if the exponent p is not large.

Namely, the following theorem is the main result of the section.

Theorem 4.1. *Let the exponent $0 \leq p < \frac{4}{3}$. Then, for every $u_0 \in L^2([-1, 1])$ with zero mean, problem (4.1) possesses a unique solution defined for all $t \geq 0$ and the following estimate holds:*

$$\|u(t)\|_{L^2} \leq Q(\|u_0\|_{L^2})e^{-\alpha t} + C_*, \quad (4.2)$$

where the positive constants C_* and α and the monotone increasing function Q are independent of t and u_0 .

Proof. Since the local existence and uniqueness theorem for Eq. (4.1) is standard and immediate, we only need to verify the dissipative estimate (4.2). As usual, we start with the case of odd periodic solutions and use the following lemma which is in fact proved in Ref. 4.

Lemma 4.2. *For every sufficiently large N there exists a 2-periodic function ϕ with zero mean such that*

$$\|\phi\|_{H^2} \leq CN^{3/2}, \quad \|\phi\|_{L^\infty} \leq CN \quad (4.3)$$

with constant C independent of N , such that, for every $u \in H^2(\Omega)$ with $u(0) = 0$,

$$\|u_{xx}\|_{L^2}^2 - (\phi_x, |u|^2) \geq N \|u\|_{L^2}^2. \quad (4.4)$$

The function ϕ with the desired properties is constructed up to scaling in Ref. 4. Indeed, it is proved there that, for any sufficiently large L , there exists a $2L$ -periodic function $\psi \in H^2(-L, L)$ such that

$$\|\psi\|_{H^2} \leq CL^{3/2}, \quad \|\psi\|_{L^\infty} \leq CL$$

and, for any $u \in H^2(-L, L)$ with $u(0) = 0$, the following inequality is satisfied:

$$\int_{-L}^L ([\partial_x^2 u(x)]^2 - u^2(x) \partial_x \psi) dx \geq \frac{1}{2} \int_{-L}^L u^2(x) dx.$$

Scaling $x = Ly$, $\phi = L^3 \psi$ and $N = CL^4$, we end up with (4.4).

We now return to the key *a priori* estimate (4.2) for the odd periodic solutions of (4.1). To this end, for any large N , we multiply Eq. (4.1) by $v := u - \phi$ where $\phi = \phi_N$ is constructed in Lemma 4.2. Then, after some transformations, we get

$$\partial_t \|v\|_{L^2}^2 + 2\|\partial_x^2 u\|_{L^2}^2 - (\partial_x \phi, |u|^2) = -2(u|u|^p, \partial_x^2 u - \partial_x^2 \phi) + 2(\partial_x^2 u, \partial_x^2 \phi). \quad (4.5)$$

Using the Cauchy-Schwartz inequality together with (4.3) and (4.4), we get

$$\partial_t \|v\|_{L^2}^2 + 1/2 \|\partial_x^2 u\|_{L^2}^2 + N \|u\|_{L^2}^2 \leq \|u\|_{L^{2(p+1)}}^{2(p+1)} + CN^3. \quad (4.6)$$

Note also that, due to Lemma 4.2, we see that, for every $q \in [1, \infty]$,

$$\|u\|_{L^q} - CN \leq \|v\|_{L^q} \leq \|u\|_{L^q} + CN, \quad (4.7)$$

where the constant C is independent of N .

Applying the interpolation inequality

$$\|u\|_{L^{2(p+1)}}^{2(p+1)} \leq C \|u\|_{L^2}^{\frac{3p+4}{2}} \|\partial_x^2 u\|_{L^2}^{\frac{p}{2}} \leq \frac{1}{2} \|\partial_x^2 u\|_{L^2}^2 + C \|u\|_{L^2}^{\frac{3p+4}{4-p}}$$

to estimate the right-hand side of (4.6), we end up with

$$\partial_t \|v\|_{L^2}^2 + N \|u\|_{L^2}^2 \leq C \|u\|_{L^2}^{\frac{3p+4}{4-p}} + CN^3. \quad (4.8)$$

In order to derive the desired dissipative estimate for u from (4.8), we will use Lemma 2.4. To this end, we scale time $t = N^2 \tau$ and introduce $\varepsilon = 1/N$. Then, (4.8) reads

$$\partial_\tau \|v\|_{L^2}^2 + \varepsilon \|u\|_{L^2}^2 \leq C \varepsilon^2 \|u\|_{L^2}^{\frac{3p+4}{4-p}} + C \varepsilon^{-1}.$$

Integrating this inequality in time, introducing $\Psi(t) := \|u(t)\|_{L^2}^2$ and using (4.7), we end up with

$$\Psi(t) \leq \Psi(0)e^{-\varepsilon t} + C \varepsilon^2 \int_0^t e^{-\varepsilon(t-s)} [\Psi(s)]^{\frac{3p+4}{4-p}} ds + C \varepsilon^{-2}, \quad (4.9)$$

where C is independent of $\varepsilon \rightarrow 0$. Equation (4.9) is the form of (2.9) with $\alpha = 2$, $\beta = \frac{3p+4}{4-p}$ and $\gamma = 1$ and the condition (2.6) reads

$$\frac{4p}{4-p} < \frac{1}{2}.$$

This condition is satisfied if and only if $p < \frac{4}{9}$. Thus, due to Lemma 2.4, the *odd* solutions of (4.1) cannot blow up in finite time and satisfy the dissipative estimate (4.2) if $p < \frac{4}{9}$, so in the particular case of odd initial data, Theorem 4.1 is proved.

We are now ready to consider the general case of periodic $u \in \dot{H}_{per}$ with using the so-called Goodman trick, see Ref. 14. Namely, we consider a *circle* of shifted functions $\phi_s(x) := \phi(x + s)$, $s \in \mathbb{R}$ and introduce

$$R(t) := \min_{s \in \mathbb{R}} \|u(t) - \phi_s\|^2.$$

Then, due to (4.3), we have the analogue of (4.7):

$$\|u(t)\|_{L^2}^2 - CN \leq R(t) \leq \|u(t)\|_{L^2}^2 + CN \quad (4.10)$$

and for the minimizer $\phi_{s(t)}$, we have

$$(u(t), \partial_x \phi_{s(t)}) \equiv 0 \quad (4.11)$$

and, at least formally (see Ref. 14 for the justification),

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} R(t) &= \frac{1}{2} \frac{d}{dt} \|u(t) - \phi_{s(t)}\|_{L^2}^2 = (\partial_t u(t), u(t) - \phi_{s(t)}) - \\ &\quad - s'(t) (\partial_x \phi_{s(t)}, u(t) - \phi_{s(t)}) = (\partial_t u(t), u(t) - \phi_{s(t)}). \end{aligned}$$

Thus, multiplying Eq. (4.1) by $2(u(t) - \phi_{s(t)})$ and arguing as before, we get

$$\frac{d}{dt} R(t) + 3/2 \|\partial_x^2 u\|_{L^2}^2 + (\partial_x \phi_{s(t)}, |u|^2) \leq \|u\|_{L^{2(p+1)}}^{2(p+1)} + CN^3 \quad (4.12)$$

However, in contrast to the odd case, we cannot apply directly Lemma 4.2 since the condition $u(t, s(t)) = 0$ is not necessarily satisfied. So, we need to introduce a time dependent “constant” $c(t) := u(t, s(t))$ and a function $w(t) = u(t) - c(t)$ for which the conditions of Lemma 4.2 are satisfied, and we have

$$\|\partial_x^2 w\|_{L^2}^2 + (\partial_x \phi_{s(t)}, |w|^2) \geq N \|w\|^2.$$

Note that, due to the zero mean condition on u ,

$$\|w\|^2 = \|u\|_{L^2}^2 + |\Omega|c(t)^2 \geq \|u\|^2$$

and, due to the orthogonality condition (4.11),

$$(\partial_x \phi_{s(t)}, |w|^2) = (\partial_x \phi_{s(t)}, |u|^2) - 2c(t)(\partial_x \phi_{s(t)}, u(t)) + c(t)^2(\partial_x \phi_{s(t)}, 1) = (\partial_x \phi_{s(t)}, |u|^2).$$

Therefore,

$$\|\partial_x^2 u\|_{L^2}^2 + (\partial_x \phi_{s(t)}, |u|^2) = \|\partial_x^2 w\|_{L^2}^2 + (\partial_x \phi_{s(t)}, |w|^2) \geq N\|u\|^2$$

and (4.12) implies that

$$\frac{d}{dt}R(t) + N\|u\|_{L^2}^2 \leq C\|u\|_{L^2}^{\frac{2(3p+4)}{4-p}} + CN^3. \quad (4.13)$$

Finally, integrating (4.13) in time, using (4.10) and arguing exactly as in the case of odd initial data, we derive the desired dissipative estimate (4.2) for general $u_0 \in \dot{H}_{per}$ and finish the proof of the theorem. $\square$

V. RELATED PROBLEMS

In this section, we apply the above considered methods to some equations which are, in a sense, close to the Kuramoto-Sivashinsky and Cahn-Hilliard equations, such as Kolmogorov-Sivashinsky-Spiegel equations and for the sixth order convective CH equations.

A. Convective CH equation vs Kolmogorov-Sivashinsky-Spiegel equation

We consider now the problem

$$\begin{cases} \partial_t u + \partial_x^4 u + \partial_x^2(2u - \delta u^3) + u \partial_x u = 0, \\ u|_{t=0} = u_0 \end{cases} \quad (5.1)$$

in the domain $\Omega := (-L, L)$ endowed by the periodic boundary conditions. We are going to show that the estimate obtained in Ref. 11 for the size of the absorbing ball can be improved at least for small values of δ where one expects that the Kuramoto-Sivashinsky dynamics will dominate. Indeed, to the best of our knowledge, all previous methods of obtaining the dissipative estimates for this equation utilize only the dissipativity which comes from the cubic term ignoring the extra dissipation provided by the convective term. As a result, the obtained estimates were *divergent* as $\delta \rightarrow 0$, see Refs. 6–8, and 11. In particular, the radius of the absorbing ball in $\dot{H}_{per}$ constructed in Ref. 6 behaves like $L^3 \delta^{-1/2}$ as $\delta \rightarrow 0$.

Using below the technique related to the Kuramoto-Sivashinsky equation (analogous to what is used in Sec. IV, we show that the radius of the absorbing ball remains bounded as $\delta \rightarrow 0$. Since we are not interested in the dependence of this radius on L , we set $L = 1$ for simplicity. Then, the following theorem holds.

Theorem 5.1. *Let $L = 1$. Then, for every $u_0 \in \dot{H}_{per}$, problem (4.1) possesses a unique solution and the following estimate holds:*

$$\|u(t)\|_{L^2}^2 \leq C\|u_0\|^2 e^{-\alpha t} + C_*, \quad (5.2)$$

where the positive constants C , α , and C_* are independent of $\delta \rightarrow 0$.

Proof. The existence and uniqueness for that equation is well-known, see, e.g., Ref. 8, so we only derive the uniform estimate (5.2). Analogously to Sec. IV, we start with the case of odd initial data u_0 and multiply (5.1) by $v = u - \varphi$ where $\varphi = \varphi_N$ is the same as in Lemma 4.2 and N will be

fixed below. Then, after the obvious transformations, we have

$$\begin{aligned} \partial_t \|v\|_{L^2}^2 + 2\|\partial_x^2 u\|_{L^2}^2 - (\partial_x \phi, |u|^2) + 6\delta(u^2, |\partial_x u|^2) = \\ = 6\delta(u^2 \partial_x u, \partial_x \phi) + 4\|\partial_x u\|_{L^2}^2 - 4(\partial_x u, \partial_x \phi) + 2(\partial_x^2 u, \partial_x^2 \phi). \end{aligned} \quad (5.3)$$

We estimate the first term in the right-hand side of (5.3) via the Cauchy-Schwartz inequality:

$$\begin{aligned} 6\delta(u^2 \partial_x u, \partial_x \phi) = 6\delta(u \partial_x u, u \partial_x \phi) \leq 6\delta(u^2, |\partial_x u|^2) + 6\delta(|\partial_x \phi|^2, u^2) \leq \\ \leq 6\delta(u^2, |\partial_x u|^2) + CN^3 \delta \|u\|_{L^2}^2, \end{aligned} \quad (5.4)$$

where we have used that $\|\partial_x \phi\|_{L^\infty} \leq C \|\phi\|_{H^2} \leq CN^{3/2}$. The rest terms in the right-hand side can be estimated in a standard way using the interpolation inequality $\|w\|_{H^1} \leq C \|w\|_{H^2}^{1/2} \|w\|_{L^2}^{1/2}$ and the Cauchy-Schwartz inequality. Therefore, using again (4.3), we see that for sufficiently large N ,

$$\partial_t \|v\|_{L^2}^2 + (\beta N - \alpha(1 + \delta N^3)) \|u\|_{L^2}^2 \leq CN^3, \quad (5.5)$$

where the positive constants β , α , and C are independent of δ and N . Finally, fixing N such that $\beta N \geq 3\alpha$, we see that, for δ being small enough that $\delta N^3 \leq 1$, the following inequality holds:

$$\partial_t \|v\|_{L^2}^2 + \alpha \|u\|_{L^2}^2 \leq CN^3$$

and the Gronwall inequality applied to it gives the desired uniform estimate (5.2). Thus, the theorem is proved in the particular case of odd initial data. The general case can be reduced to that particular one using the Goodman trick again, exactly as in Sec. IV. So, the theorem is proved.

Remark 5.2. It looks natural to consider the mixture of problems (4.1) and (5.1), namely,

$$\partial_t u + uu_x + (u_{xx} + u|u|^p - \delta u^3)_{xx} = 0$$

with $p < \frac{4}{9}$ and study the limit $\delta \rightarrow 0$. However, the above method does not work at least directly in this case (at least without stronger assumptions on p). Indeed, as we see from (5.5), if we allow N to be dependent on the norm of the initial data (as in Sec. IV), we also need to decrease δ in the dependence on the initial data. Thus, the problem of obtaining the uniform with respect to $\delta \rightarrow 0$ estimates for that case remains open.

B. Sixth order convective Cahn-Hilliard equation

We consider here the following sixth order convective Cahn-Hilliard equation

$$\begin{cases} \partial_t u - \partial_x^4 (\partial_x^2 u + u - u^3) + u \partial_x u = 0, \\ u|_{t=0} = u_0 \end{cases} \quad (5.6)$$

on a bounded interval $\Omega = (-L, L)$. This equation was derived in Ref. 27 as a model of process of growing crystalline surface with small slopes that undergoes faceting where $u = \partial_x h$ is the slope of a surface $h(x, t)$. In Ref. 17 the authors proved existence and uniqueness of the global solution to initial boundary value problem for (5.6) under periodic boundary conditions.

We are going to show that the semigroup generated by the initial boundary value problem for (5.6) under the *periodic* boundary conditions is dissipative (the case of, say, Dirichlet boundary conditions can be treated similarly), namely, the following proposition holds.

Proposition 5.3. For any $u_0 \in \dot{H}_{per}^{-1}$, problem (5.6) possesses a unique solution $u \in C(\mathbb{R}_+, \dot{H}_{per}^{-1}) \cap L_{loc}^2(\mathbb{R}_+, \dot{H}_{per}^2)$ and the following dissipative estimate holds:

$$\|u(t)\|_{\dot{H}_{per}^{-1}} \leq Q(\|u_0\|_{\dot{H}_{per}^{-1}}) e^{-\alpha t} + C_*, \quad (5.7)$$

for the positive constants α and C_* and monotone function Q which are independent of t and u_0 .

Proof. Applying to both sides of (5.6) the operator $P = (-\partial_x^2)^{-1}$ we get an equivalent problem

$$\begin{cases} P \partial_t u + \partial_x^2 (\partial_x^2 u + u - u^3) - P(u \partial_x u) = 0, \\ u|_{t=0} = u_0. \end{cases} \quad (5.8)$$

Multiplying Eq. (5.8) by u and integrating over $(0, L)$ we obtain

$$\frac{1}{2} \frac{d}{dt} \|P^{\frac{1}{2}} u\|^2 - \left(P^{\frac{1}{2}} (u \partial_x u), P^{\frac{1}{2}} u \right) + \|\partial_x^2 u\|^2 - \|\partial_x u\|^2 + 3(u^2, (\partial_x u)^2) = 0.$$

Employing the inequality (2.1) and the interpolation inequality

$$\|\partial_x u\|^2 \leq \varepsilon_1 \|P^{\frac{1}{2}} u\| + C_1 \|\partial_x^2 u\|,$$

which is valid for each $u \in \dot{H}_{per}^2$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|P^{\frac{1}{2}} u\|^2 + (1 - \varepsilon_1) \|\partial_x^2 u\|^2 + 3(u^2, (\partial_x u)^2) &\leq d_0 \|u^2\| \|P^{\frac{1}{2}} u\| + C_1 \|P^{\frac{1}{2}} u\|^2 \leq \\ \varepsilon_2 \|u^2\|^2 + (C_1 + C_2) \|P^{\frac{1}{2}} u\|^2 &\leq 2\varepsilon_2 \|u^2\|^2 + \frac{[d_0(C_1 + C_2)]^2}{2\varepsilon_2} L. \end{aligned} \quad (5.9)$$

Finally we use the inequality (which follows from (2.1))

$$\|u^2\|^2 \leq 4d_0^2 (u^2, (\partial_x u)^2)$$

and obtain from (5.9) the following estimate:

$$\frac{1}{2} \frac{d}{dt} \|P^{\frac{1}{2}} u\|^2 + (1 - \varepsilon_1) \|\partial_x^2 u\|^2 + (3 - 8\varepsilon_2 d_0^2) (u^2, (\partial_x u)^2) \leq K_0. \quad (5.10)$$

By choosing $\varepsilon_1 = \frac{1}{2}$ and $\varepsilon_2 = \frac{3}{8d_0^2}$ and applying the Gronwal inequality, we deduce (5.7) and finish the proof of the proposition. $\square$

Remark 5.4. Using the standard parabolic regularity, one may show that the solution $u(t)$ constructed in Proposition 5.3 becomes C^∞ (and even Gevrey) regular for all $t > 0$.

Remark 5.5. By using the same arguments as in the proof of Theorem 3.1 we can show that a wide class of solutions to the initial boundary value problem for the sixth order convective CH equations with concave potential

$$\partial_t u - \partial_x^4 (\partial_x^2 u + u + u^3) + u \partial_x u = 0 \quad (5.11)$$

blow up in a finite time.

We would like also note that the blow up theorem for sixth order unstable CH equations (5.11) *without* the convective term can be established by using the concavity method of Levine since in this situation the equation can be written in the form (1.3).

Finally, arguing as in the proof of Theorem 4.1, one can show the global existence and dissipativity of solutions of the following problem:

$$\partial_t u - \partial_x^4 (\partial_x^2 u + u + u|u|^p) + u \partial_x u = 0 \quad (5.12)$$

if $p < p_0$ for some exponent $p_0 > 0$, so the presence of the convective term prevents blow up in that situation as well. However, in order to compute the exponent p_0 , we need the analog of the sharp Lemma 4.2 for the six order operator which we do not present here.

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